Spacelike Surfaces of Constant Mean Curvature having Continuous Internal Symmetry in Minkowski three space

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MSc Thesis by Research
Submitted in fulfilment of the requirements
for the degree of Master of Science in Applied Mathematical Sciences
at Dublin City University, 1995

I hereby certify that this material, which I now submit for assessment on the programme of study leading to the award of Master of Science in Applied Mathematical Sciences is entirely my own work and has not been taken from the work of others save and to the extent that such work has been cited and acknowledged within the text of my work

Signed

Candidate

Date 7 December 1995

Acknowledgements

I wish to acknowledge all of the following

My supervisor· Dr Michael Clancy, *mens sana in corpore sano* - I shall miss the yap and the cup of tea at our meetings Thank you for everything you've done for me

The Department. Thanks also to the other members of the D C U mathematics department, both past and present, for all their time and friendliness

My friends. To list all I would wish to thank would be futile and so I'll just say a special thanks to my fellow mathematics postgrads Kieran Arthur, Martin O Gorman, Mary Dunne, Fawzi Abdelwahid, Paul Farrell, Sinead Breen, Tony O'Neill ,Brian Murphy and in particular to my closest friends Mark Ward and Kieran Murphy

My family: To my folks who are simply always there - for every aid, financial and otherwise, and for being mad as hatters to boot I also wish to thank the other members of my family for their support and for all the craic we had over the years

Remark

I wish to thank the external examiner Prof D Simms for making following suggestions which lead to simplification and unification in the presentation of chapter two

As given on page 9, embedd $\mathbb{R}^3$in $\mathbb{C}^3$ vis

$$\dagger \quad \mathbb{R}^3 \to \mathbb{C}^3 \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto x^\dagger = \begin{pmatrix} x_1 \\ x_2 \\ \imath x_3 \end{pmatrix} \quad \text{then}$$

1 $\langle , \rangle_1 = \langle x^\dagger, y^\dagger \rangle$

2 $x^\dagger \times y^\dagger = \imath(x * y)^\dagger$, where $\times$ is the usual crossproduct and $*$ is defined as on page 7

3 $\dagger$ leads to the embedding of $GL(3, \mathbb{R}) \hookrightarrow GL(3, \mathbb{C})$ by the commutative diagram

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "L"] \arrow[d, hook, "\dagger"'] & \mathbb{R}^3 \arrow[d, hook, "\dagger"] \\ \mathbb{C}^3 \arrow[r, "L^\dagger"] & \mathbb{C}^3 \end{tikzcd}$$

i e $L^\dagger x^\dagger = (Lx)^\dagger$ So if all linear maps are represented relative to the basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(both in $\mathbb{R}^3$and $\mathbb{C}^3$) then in matrix notation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^\dagger = \begin{pmatrix} a_{11} & a_{12} & -\imath a_{13} \\ a_{21} & a_{22} & -\imath a_{23} \\ \imath a_{31} & \imath a_{32} & a_{33} \end{pmatrix}$$

as given on page 9

4 $SO(2,1)$ is then replaced by $SO(2,1)^\dagger \subset SO(3,\mathbb{C})$

5 If we now identify $\mathbb{R}^3$, $SO(2,1)$ etc with their embedded images, i e write x to mean $x^\dagger$, A to mean A^T etc then the adjoint of A is just $A^\dagger$ So the Lie algebra condition for $SO(2,1)$ in $SO(3,\mathbb{C})$ is just

$$A + A^T = 0$$

from which lemma 2 4 follows immediately

Contents

Abstract

The purpose of this thesis is to give a characterization of all spacelike constant mean curvature surfaces which have continuous internal symmetry in Minkowski three space The properties that these surfaces must satisfy leads to a system of partial differential equations and every solution of this system results in a desired surface Further examination of this system leads to a differential relation that the metric must satisfy The behaviour of the solution to this relation is investigated to determine if the resulting surface is complete

Chapter 1

Introduction

In 1841 Delaunay [2] charterized constant mean curvature surfaces of revolution (in Euclidean three space) as those whose profile curve is the roulette of a conic

Theorem 1.1 *Delaunay*

A curve γ in the x-y plane generates a surface of constant mean curvature when rotated about the $x-$axis if and only if γ is a piece of the roulette of a conic i e the locus of the focus of a conic in this plane as it is rolled along the $x-$axis

These surfaces admit a one parameter group of internal isometries This was generalized by Smyth in [6] and the result was

Theorem 1 2 *Smyth*

For each integer $m \geq 0$ there exists a one-parmeter family of conformal immersions

$$f_m \quad \mathbb{C} \rightarrow \mathbb{R}^3$$

with constant mean curvature 1, such that the induced metric is complete and invariant by the group of rotations about 0 Moreover 0 is an umbilic of index $-m/2$, only powers of the rotation through $2\pi/(m+2)$ about 0 extend to motions of $\mathbb{R}^3$ and the associates of f_m are given by $(f_m)_\theta = f_m \circ e^{-i\theta}$ Conversly any complete surface of constant mean curvature 1 admitting a one parameter group of isometries is, to within associates, congruent either to such an f_m or to a Delaunay surface

In this classification it is assumed that M is simply connected and it is shown that (M, g) is conformally equivalent to the region in $\mathbb{C}$ $a < Re(z) < b$ where a and b are constant (either finite or infinite) and the metric $g = e^{\phi}|dz|^2$ is invariant by translations in the y-directions An alternative characterization to Smyth's was given by Burns and Clancy [1] whose result is as follows

Theorem 1.3 *Burns and Clancy*

If $M = \{z \in \mathbb{C} \quad a < Re z < b\}$ *and if* $g = \lambda^2(dx^2 + dy^2)$, *then* $f \quad (M, g) \to \mathbb{R}^3$ *is an isometric immersion of constant mean curvature* H *if and only if* f *satisfies the following system of p d e 's*

$$\begin{aligned} f_{xx} &= -\alpha f_x - (\mathbf{E} - \alpha H f) \times f_y + 2H f_x \times f_y \\ f_{xy} &= -\alpha f_y + (\mathbf{E} - \alpha H f) \times f_x \\ f_{yy} &= \quad \alpha f_x + (\mathbf{E} - \alpha H f) \times f_y \end{aligned}$$

with initial conditions

$$\|f_x(x_0, x_0)\| = \|f_y(x_0, x_0)\| = \lambda(x_0) \quad \text{and} \quad \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle = 0$$

where $\times$ *denotes the usual cross product in* $\mathbb{R}^3, \alpha$ *is an arbitary (non-negative) real constant and* $\mathbf{E}$ *is an arbitary constant vector in* $\mathbb{R}^3$

In this thesis we follow the arguments set out by Burns and Clancy to find a classification of spacelike constant mean curvature surfaces with continuous internal symmetry in Minkowski three space We note that Minkowski three space is just $\mathbb{R}^3$ with the scalar product,$\langle , \rangle_1$, between two vectors $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$ being defined as

$$\langle x, y\rangle_1 = x_1 y_1 + x_2 y_2 - x_3 y_3$$

The main theorems are as follows

Theorem 1.4

Let $M = \{\mathbb{C} |\ x_1 < Re z < x_2\}$ where x_1, x_2 are constants (either finite or infinite) and $g = e^{\phi(x)}(dx^2 + dy^2)$ Then every immersion f $(M, g) \to (\mathbb{R}^3, \langle , \rangle_1)$ which satisfies either conditions 1 or 2 given below represents a spacelike surface in $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ and the maps Ψ_t $(x, y) \to (x, y + t)$ form a one-parameter group of internal symmetries Conversely every spacelike surface in $\mathbb{R}^3$ with continuous internal symmetry and constant mean curvature $H \neq 0$ arises in this way

1

$$\begin{aligned} f_{xy} &= f_y + Hf * f_x \\ f_{yy} &= -f_x + Hf * f_y \\ f_{xx} + f_{yy} &= 2Hf_x * f_y \end{aligned}$$

2

$$\begin{aligned} f_y &= \mathbf{E} * f \\ \xi_y &= \mathbf{E} * \xi \\ f_{xx} + f_{yy} &= 2Hf_x * f_y \end{aligned}$$

where $\mathbf{E} \in \mathbb{R}^3$

Some work has been covered with regard to the completeness of these surfaces but only partial results were found These can be seen in chapter 12

Theorem 1 5

Every Spacelike Minimal Surface (i e spacelike surface of constant mean curvature equal to zero) with continuous internal symmetry in Minkowski three space is, up to a hyperbolic motion, one of the following

1

$$f(x,y) = \frac{lr_1}{6}(x+r_2)\begin{pmatrix} 3/r_1^2 + (x+r_2)^2 - 3y^2 \\ -6/ry \\ -3/r_1^2 + (x+r_2)^2 - 3y^2 \end{pmatrix} \tag{1 1}$$

where $l, r_1, r_2 \in \mathbb{R}$

2

$$f(x,y) = r_1 \begin{pmatrix} \sinh(-\epsilon y)\cos(\epsilon x + r_2) \\ \epsilon(x+r_2) \\ \cosh(-\epsilon y)\cos(\epsilon x + r_2) \end{pmatrix} \tag{1 2}$$

where $r_1, r_2 \in \mathbb{R}$

3

$$f(x,y) = -r_1 \begin{pmatrix} \sinh(\epsilon x + r_2)\cos y \\ -\sinh(\epsilon x + r_2)\sin y \\ -(\epsilon x + r_2) \end{pmatrix} \tag{1 3}$$

where $r_1, r_2 \in \mathbb{R}$

4

$$f(x,y) = r_1 e^{-\alpha x} \begin{pmatrix} \frac{a_1}{r_1 \epsilon} \cos \alpha y \\ \cos \alpha y \cosh \epsilon y \sin(\epsilon x + r_2) + \sin \alpha y \sinh \epsilon y \sin(\epsilon x - r_2) \\ \cos \alpha y \sinh \epsilon y \sin(\epsilon x + r_2) + \sin \alpha y \cosh \epsilon y \sin(\epsilon x - r_2) \end{pmatrix}$$

where $a_1, r_1, r_2 \in \mathbb{R}$

5

$$f(x,y) = \frac{a_1 r_1}{2} e^{-\alpha x} \begin{pmatrix} [\frac{1}{r_1^2} + \frac{1}{\alpha^2} + (x + r_2)^2 - y^2] \cos \alpha y + 2y(x + r_2) \sin \alpha y \\ \frac{2}{r_1}(-y \cos \alpha y + (x + r_2) \sin \alpha y) \\ [\frac{-1}{r_1^2} + \frac{1}{\alpha^2} + (x + r_2)^2 - y^2] \cos \alpha y + 2y(x + r_2) \sin \alpha y \end{pmatrix}$$

where $a_1, r_1, r_2 \in \mathbb{R}$

6

$$f(x,y) = \begin{pmatrix} r_1 \cos(-\alpha + \epsilon) y e^{(-\alpha+\epsilon)x} + r_2 \cos(\alpha + \epsilon) y e^{(-\alpha-\epsilon)x} \\ -r_1 \sin(-\alpha + \epsilon) y e^{(-\alpha+\epsilon)x} - r_2 \sin(\alpha + \epsilon) y e^{(-\alpha-\epsilon)x} \\ -\frac{a_1}{\epsilon} e^{-\alpha x} \cos \alpha y \end{pmatrix}$$

where $a_1, r_1, r_2 \in \mathbb{R}$

7

$$f(x,y) = \begin{pmatrix} r_2 e^{-2\alpha x} \cos(2\alpha y) - \frac{r_1}{2\alpha} x \\ -r_2 e^{-2\alpha x} \sin(2\alpha y) + \frac{r_1}{2\alpha} y \\ -\frac{a_1}{\alpha} e^{-\alpha x} \cos(\alpha y) \end{pmatrix}$$

where $a_1, r_1, r_2 \in \mathbb{R}$

Chapter 2

The Lie Group SO(2,1)

We begin with some preliminary remarks on indefinite scalar product spaces (see, for example Nomizu [5])

Theorem 2.1

Suppose that $\langle , \rangle_1$ is a bilinear form on a real vector space $\mathbf{V}$ of dimension n Then there exists a basis $\{u_1, u_2, \quad , u_n\}$ of $\mathbf{V}$ such that

1 $\langle u_i, u_j \rangle_1 = 0$ *for* $i \neq j$

2 $\langle u_i, u_i \rangle_1 = 1$ *for* $1 \leq i \leq p$

3 $\langle u_i, u_i \rangle_1 = -1$ *for* $p + 1 \leq i \leq r$

4 $\langle u_i, u_i \rangle_1 = 0$ *for* $r + 1 \leq i \leq n$

The numbers r and p are determined solely by the bilinear form, r is called the rank, $r - p$ is called the index, and the ordered pair $(p, r - p)$ is called the signature The theorem shows that any two spaces of the same dimension with bilinear forms of the same signature are isometrically isomorphic By a scalar product we mean a nondegenerate bilinear form, i e , a form with rank equal to the dimension of $\mathbf{V}$ We let $U^\perp$ denote the orthogonal complement of a subspace U with respect to the given scalar product

Theorem 2.2

Suppose that $\langle , \rangle_1$ is a scalar product on a finite dimensional real vector space V and suppose that U is a subspace of V, then

1 $(U^\perp)^\perp = U$ *and* $\dim U + \dim U^\perp = \dim V$

2 *The form $\langle , \rangle_1$ is nondegenerate on U iff it is nondegenerate on $U^\perp$ and when it is nondegenerate on U, then $V = U \oplus U^\perp$ (the direct sum of U and $U^\perp$)*

3 *If V is the orthogonal direct sum of two subspaces U and W, then the form is nondegenerate on U and W, and $W = U^{\perp}$*

Let $\langle , \rangle_1$ be the indefinite scalar product on $\mathbb{R}^n$ defined by

$$\langle x, y\rangle_1 = x_1y_1 + x_2y_2 + \quad + x_{n-1}y_{n-1} - x_ny_n,$$

where $x = (x_1, x_2, \quad , x_n)^T \in \mathbb{R}^n$ and $y = (y_1, y_2, \quad , y_n)^T \in \mathbb{R}^n$ We call this space Minkowski n-space and the scalar product $\langle , \rangle_1$ shall be called the Minkowski metric A vector x is said to be spacelike, timelike, or lightlike depending on whether $\langle x, x\rangle_1$ is positive, negitive or zero, respectively In Minkowski n-space the set of all lightlike vectors, given by the equation

$$x_1^2 + x_2^2 + \quad + x_{n-1}^2 = x_n^2,$$

forms a cone of revolution, called the light cone Timelike vectors are "inside the cone" and spacelike vectors are "outside the cone"

If x is a non-zero vector, let $x^{\perp}$ denote the orthogonal complement of x with respect to the Minkowski metric If x is timelike, then the metric restricts to a positive definite form on $x^{\perp}$, and $x^{\perp}$ intersects the light cone only at the origin If x is spacelike, then the metric has signature $(n-1, 1)$ on $x^{\perp}$, and $x^{\perp}$ intersects the cone in a cone of one dimension less If x is lightlike, then $x^{\perp}$ is tangent to the cone along the line through the origin determined by x The metric has signature $(n-1, 0)$ on this $n-1$-dimensional plane

Now, for all $x, y \in \mathbb{R}^3$ we define

$$x * y = \begin{pmatrix} x_3y_2 & - & x_2y_3 \\ x_1y_3 & - & x_3y_1 \\ x_1y_2 & - & x_2y_1 \end{pmatrix}$$

Remark

$x * y$ is just the usual cross product $x \times y$ with the first two components negated. It can easily be verified that the following conditons hold

- $x * y = -y * x$
- $x * (y + z) = x * y + x * z$
- $(x + y) * z = x * z + y * z$
- For every $r \in \mathbb{R}$, $rx * y = (rx) * y = x * (ry)$
- $x * (y * z) + y * (z * x) + z * (x * y) = 0$

So that $(\mathbb{R}^3, *)$ is a Lie Algebra with bracket product

$$[x, y] = x * y$$

As usual for all $x \in \mathbb{R}^3$ we define $ad\, x \in \mathcal{E}nd(\mathbb{R}^3)$, the endomorphisms of $\mathbb{R}^3$, by

$$(ad\, x)y = [x, y]$$

Then the matrix representation of $ad\, x$ relative to the standard basis for $\mathbb{R}^3$ is

$$ad\, x = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$$

Orthogonal Groups

Definition :

$$O(2,1) = \{ \text{ linear } \Theta \quad \mathbb{R}^3 \to \mathbb{R}^3 \mid \langle \Theta x, \Theta y \rangle_1 = \langle x, y \rangle_1 \quad \forall\, x, y \in \mathbb{R}^3 \}$$

$$\begin{aligned} SO(2,1) &= \text{connected component to the identity of } O(2,1) \\ &= \{ \Theta \in O(2,1) \mid \det\Theta = 1 \text{ and } sign\langle e_3, \Theta e_3 \rangle_1 = -1 \} \end{aligned}$$

These are Lie Groups in the usual way The Lie Algebra of $SO(2,1)$ is given by

$$\mathcal{SO}(2,1) = \{3 \times 3 \text{ matrices } A \mid e^A \in SO(2,1)\},$$

where e^A is the usual exponential of the matrix A To obtain a more explicit description of $\mathcal{SO}(2,1)$ observe that $A \in \mathcal{SO}(2,1)$

$$\begin{aligned}
\Rightarrow \qquad \langle e^{tA}x, e^{tA}y\rangle_1 &= \langle x, y\rangle_1 \quad \forall x, y \in \mathbb{R}^3 \ , t \in \mathbf{R} \\
\frac{d}{dt}\langle e^{tA}x, e^{tA}y\rangle_1\Big|_{t=0} &= 0 \\
\langle Ax, y\rangle_1 + \langle x, Ay\rangle_1 &= 0 \qquad (2\,1)
\end{aligned}$$

That is, A is skew-symmetric relative to $\langle,\rangle_1$ Thus, it is clear that A being skew-symmetric is a necessary and sufficient conditon for $A \in \mathcal{SO}(2,1)$

For any $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, define $x^\dagger = (x_1, x_2, \imath x_3)^T \in \mathbb{C}^3$ where $\imath = \sqrt{-1}$ and T denotes transpose, then

$$\langle x, y\rangle_1 = x_1y_1 + x_2y_2 - x_3y_3 = \left(x^\dagger\right)^T \ y^\dagger$$

For

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

define

$$A^\dagger = \begin{pmatrix} a_{11} & a_{12} & -\imath a_{13} \\ a_{21} & a_{22} & -\imath a_{23} \\ \imath a_{31} & \imath a_{32} & a_{33} \end{pmatrix},$$

then

$$A^\dagger \, x^\dagger = (A\,x)^\dagger$$

and

$$\left(A^{\dagger}\right)^{T} = \left(\left(A^{\dagger\dagger}\right)^{T}\right)^{\dagger}$$

At this stage it is worth alerting the reader to the remark made on page 11

Lemma 2 3

If A is a 3×3 matrix and $x, y \in \mathbb{R}^3$, then

$$\langle Ax, y\rangle_1 = \langle x, \left(A^{\dagger\dagger}\right)^{T} y\rangle_1$$

Proof :

$$\begin{aligned}
\langle Ax, y\rangle_1 &= \left((Ax)^{\dagger}\right)^{T} y^{\dagger} = \left(A^{\dagger}x^{\dagger}\right)^{T} y^{\dagger} \\
&= (x^{\dagger})^{T}\left(A^{\dagger}\right)^{T} y^{\dagger} = (x^{\dagger})^{T}\left(A^{\dagger}\right)^{T} y^{\dagger} \\
&= (x^{\dagger})^{T}\left(\left(A^{\dagger\dagger}\right)^{T}\right)^{\dagger} y^{\dagger} = (x^{\dagger})^{T}\left(\left(A^{\dagger\dagger}\right)^{T} y\right)^{\dagger} \\
&= \langle x, \left(A^{\dagger\dagger}\right)^{T} y\rangle_1
\end{aligned}$$

Lemma 2.4

$\mathcal{SO}(2,1)$ is the set of all 3×3 real matrices A of the form

$$A = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

Proof .

Using Lemma 2 3 and (2 1) we see that if $A \in \mathcal{SO}(2,1)$ then

$$A = -(A^{\dagger\dagger})^T$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{21} & a_{31} \\ -a_{12} & -a_{22} & a_{32} \\ a_{13} & a_{23} & -a_{33} \end{pmatrix}$$

and hence we have that

$$a_{11} = a_{22} = a_{33} = 0$$

and

$$a_{21} = -a_{12}, \quad a_{31} = a_{13}, \quad a_{32} = a_{23}$$

Thus if $A \in \mathcal{SO}(2,1)$

$$A = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \text{ for some } a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$$

which is the matrix representation of $ad\, a$, for some a relative to the standard basis for $\mathbb{R}^3$ Thus $ad \quad \mathbb{R}^3 \to \mathcal{SO}(2,1) \quad x \to ad\, x$ is a Lie Algebra isomorphism We note that

$$\langle (ad\, n)x, y\rangle_1 = -\langle x, (ad\, n)y\rangle_1$$

and

$$\begin{aligned} \langle x * y, z\rangle_1 &= \langle -y * x, z\rangle_1 = \langle -(ad\, y)x, z\rangle_1 \\ &= \langle x, (ad\, y)z\rangle_1 = \langle x, y * z\rangle_1 \end{aligned}$$

Remark

Both x and y are orthogonal to $x * y$, because

$$\langle x * y, y\rangle_1 = \langle x, y * y\rangle_1 = 0$$

and similarly
$$\langle x * y, x\rangle_1 = 0$$

Lemma 2.5

For $x, y, z \in \mathbb{R}^3$
$$x * (y * z) = \langle x, y\rangle_1 z - \langle x, z\rangle_1 y$$

Proof :

If y and z are linearly dependent then the result is trivial as both sides are zero We now fix y and z and assume they are linearly independent The vector $x * (y * z)$ is orthogonal to $y * z$ and therefore lies in the plane spanned by y and z, accordingly,

$$x * (y * z) = \alpha_{yz}(x)y + \beta_{yz}(x)z \qquad (2\ 2)$$

for some $\alpha_{yz}(x), \beta_{yz}(x) \in \mathbf{R}$ Also the map $\mathbb{R}^3 \to \mathbb{R}^3 \quad x \to x * (y * z)$ is linear, hence the maps

$$\mathbb{R}^3 \to \mathbf{R} \quad x \to \alpha_{yz}(x)$$

$$\mathbb{R}^3 \to \mathbf{R} \quad x \to \beta_{yz}(x)$$

are linear, so there exists an A,B $\in \mathbb{R}^3$, which depend on y and z but not x, such that

$$\alpha_{yz}(x) = \langle A, x\rangle_1 \text{ and } \beta_{yz}(x) = \langle B, x\rangle_1$$

and so

$$x * (y * z) = \langle A, x\rangle_1 y + \langle B, x\rangle_1 z \ \forall x \in \mathbb{R}^3 \qquad (2\ 3)$$

Setting $x = y * z$ in (2 3) , we obtain

$$0 = (y * z) * (y * z) = \langle A, y * z\rangle_1 y + \langle B, y * z\rangle_1 z$$

Hence $\langle A, y * z\rangle_1 = \langle B, y * z\rangle_1 = 0$ and we can write

$$A = a_1 y + a_2 z$$

$$B = b_1 y + b_2 z$$

for some $a_i, b_i \in \mathbf{R}$, $1 \leq i \leq 2$ which may depend on y and z Substitution into (2 3) gives

$$x * (y * z) = \langle a_1 y + a_2 z, x\rangle_1 y + \langle b_1 y + b_2 z, x\rangle_1 z \qquad (2\ 4)$$

and it follows that

$$0 = \langle x * (y * z), x\rangle_1 = \langle a_1 y + a_2 z, x\rangle_1 \langle y, x\rangle_1 + \langle b_1 y + b_2 z, x\rangle_1 \langle z, x\rangle_1 \qquad (2\ 5)$$

now choose $x \perp y$, $x \not\perp z$ in (2 5), where $\perp$ means "orthogonal to", then

$$\langle b_1 y + b_2 z, x\rangle_1 = 0 \Rightarrow b_2 \langle z, x\rangle_1 = 0 \Rightarrow b_2 = 0$$

choose $x \not\perp y$, $x \perp z$, in (2 5), then

$$\langle a_1 y + a_2 z, x\rangle_1 = 0 \Rightarrow a_1 \langle y, x\rangle_1 = 0 \Rightarrow a_1 = 0$$

$$\Rightarrow \qquad 0 = (a_2 + b_1)\langle z, x\rangle_1 \langle y, x\rangle_1 \;\; \forall\, x \in \mathbb{R}^3 \qquad (2\ 6)$$

if we now choose an x which is $\perp$ to neither y nor z in (2 6), we obtain

$$0 = (a_2 + b_1) \Rightarrow b_1 = -a_2$$

and substituting this result into (2 4) we obtian

$$x * (y * z) = a_2[\langle z, x\rangle_1 y - \langle y, x\rangle_1 z] \ \ \forall\, x \ \in \mathbb{R}^3 \tag{2 7}$$

Now let $x = z$ and substitute this into (2 7)

$$\begin{aligned} z * (y * z) &= a_2[\|z\|^2 y - \langle y, z\rangle_1 z] \\ \langle y, z * (y * z)\rangle_1 &= a_2[\|z\|^2\|y\|^2 - \langle y, z\rangle_1^2] \\ \langle (y * z), (y * z)\rangle_1 &= -a_2[\langle y, z\rangle_1^2 - \|z\|^2\|y\|^2] \end{aligned}$$

finally, expanding componentwise we see

$$\langle (y * z), (y * z)\rangle_1 = \|(y * z)\|^2 = \langle y, z\rangle_1^2 - \|z\|^2\|y\|^2 \tag{2 8}$$

hence $a_2 = -1$ and (2 7) gives the required result

Lemma 2.6

If the spacelike vectors $x, y \in \mathbb{R}^3$ satisfy

$$\langle x, y\rangle_1 = 0 \quad \textit{and} \quad \langle x, x\rangle_1 = \langle y, y\rangle_1 = e^{\phi} \quad \text{, for some } \phi \in \mathbb{R}$$

and if we define $\xi \in \mathbb{R}^3$ by

$$\xi = \frac{1}{e^{\phi}} x * y = e^{-\phi} \begin{pmatrix} x_3 y_2 & - & x_2 y_3 \\ x_1 y_3 & - & x_3 y_1 \\ x_1 y_2 & - & x_2 y_1 \end{pmatrix}$$

then the following statements hold

1. $\langle \xi, \xi \rangle_1 = -1$

2. the matrix $A = \left[e^{-\phi/2}x \quad e^{-\phi/2}y \quad \xi \right] \in O(2,1)$

3. $\xi * x = -y$ and $\xi * y = x$

Proof ·

1

$$\begin{aligned}
\langle \xi, \xi \rangle_1 &= e^{-2\phi} \langle x * y, x * y \rangle_1 \\
&= e^{-2\phi} (\langle x, y \rangle_1^2 - \|x\|^2 \|y\|^2) \quad \text{by (2 8)} \\
&= -e^{-2\phi} e^{\phi} e^{\phi} \\
&= -1
\end{aligned}$$

2 Since

$$A = e^{-\phi/2} \begin{pmatrix} x_1 & y_1 & e^{\phi/2}\xi_1 \\ x_2 & y_2 & e^{\phi/2}\xi_2 \\ x_3 & y_3 & e^{\phi/2}\xi_3 \end{pmatrix}$$

we have

$$\left(A^{\dagger\dagger}\right)^T = e^{-\phi/2} \begin{pmatrix} x_1 & x_2 & -x_3 \\ y_1 & y_2 & -y_3 \\ -e^{\phi/2}\xi_1 & -e^{\phi/2}\xi_2 & e^{\phi/2}\xi_3 \end{pmatrix}$$

and hence

$$\left(A^{\dagger\dagger}\right)^T A = e^{-\phi}\begin{pmatrix} x_1 & x_2 & -x_3 \\ y_1 & y_2 & -y_3 \\ -e^{\phi/2}\xi_1 & -e^{\phi/2}\xi_2 & e^{\phi/2}\xi_3 \end{pmatrix}\begin{pmatrix} x_1 & y_1 & e^{\phi/2}\xi_1 \\ x_2 & y_2 & e^{\phi/2}\xi_2 \\ x_3 & y_3 & e^{\phi/2}\xi_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= I$$

For any $u, v \in \mathbb{R}^3$

$$\langle Au, Av\rangle_1 = \langle u, \left(A^{\dagger\dagger}\right)^T Av\rangle_1 = \langle u, v\rangle_1$$

and hence $A \in O(2,1)$

$$\text{Note} \qquad \det(A) = -e^{-\phi}\langle x * y, \xi\rangle_1 = -\langle \xi, \xi\rangle_1 = 1$$

Furthermore

$$\langle Ae_3, Ae_3\rangle_1 = \langle e_3, e_3\rangle_1 = -1,$$

so that the x_3-component of Ae_3 cannot be zero Hence $\langle e_3, Ae_3\rangle_1 \neq 0$ Now

$$sign\langle e_3, Ae_3\rangle_1 = sign(\langle e_3, \xi\rangle_1) = sign(-\xi_3) = -sign(\xi_3)$$

Thus A is in the connected component of the identity (or equivalently $A \in SO(2,1)$) providing $\xi_3 > 0$ Therefore either

$$\left[e^{-\phi/2}x \quad e^{-\phi/2}y \quad x * y\right] \in SO(2,1)$$

or

$$\left[e^{-\phi/2}y \quad e^{-\phi/2}x \quad y*x\right] \in SO(2,1)$$

3

$$\begin{aligned} \xi * x &= e^{-\phi}(x*y)*x \\ &= -e^{-\phi}x*(x*y) \\ &= -e^{-\phi}(\langle x,x\rangle_1 y - \langle x,y\rangle_1 x) \\ &= -e^{-\phi}(e^{\phi}y) \quad (as\langle x,y\rangle_1 = 0) \\ &= -y \end{aligned}$$

Similarly,

$$\begin{aligned} \xi * y &= e^{-\phi}(x*y)*y \\ &= -e^{-\phi}y*(x*y) \\ &= -e^{-\phi}(\langle y,x\rangle_1 y - \langle y,y\rangle_1 x) \\ &= -e^{-\phi}(-e^{\phi}x) \quad (as\langle x,y\rangle_1 = 0) \\ &= x \end{aligned}$$

Lemma 2 7

Let $\Theta(t) \in SO(2,1)$ for all $t \in \mathbb{R}$ Then

$$\Theta(\Theta^{\dagger\dagger})^T = ad\,\eta(t)$$

where $\eta(t) \in \mathbb{R}^3$ for all $t \in \mathbb{R}$

Proof .

We need to show that

$$\left(\left\{\Theta(\Theta^{\dagger\dagger})^T\right\}^{\dagger\dagger}\right)^T = -\Theta(\Theta^{\dagger\dagger})^T$$

We first note that for all 3×3 matrices X, Y we have

$$(XY)^\dagger = X^\dagger Y^\dagger$$

and

$$(X^T)^{\dagger\dagger} = (X^{\dagger\dagger})^T$$

so that

$$\begin{aligned}
\left[\left\{\Theta(\Theta^{\dagger\dagger})^T\right\}^{\dagger\dagger}\right]^T &= \left[\Theta^{\dagger\dagger}((\Theta^{\dagger\dagger})^T)^{\dagger\dagger}\right]^T \\
&= \left[\Theta^{\dagger\dagger}((\Theta^{\dagger\dagger})^{\dagger\dagger})^T\right]^T \\
&= \left[\Theta^{\dagger\dagger}\Theta^T\right]^T \\
&= \Theta(\Theta^{\dagger\dagger})^T
\end{aligned}$$

As $\Theta \in SO(2,1)$ we have by lemma 2 3 that $\Theta(\Theta^{\dagger\dagger})^T = I$ so that $\Theta(\Theta^{\dagger\dagger})^T + \Theta(\Theta^{\dagger\dagger})^T = 0$ i e

$$\Theta(\Theta^{\dagger\dagger})^T = -\Theta(\Theta^{\dagger\dagger})^T$$

Hence

$$\left(\left\{\Theta(\Theta^{\dagger\dagger})^T\right\}^{\dagger\dagger}\right)^T = -\Theta(\Theta^{\dagger\dagger})^T$$

and the lemma is proved

Chapter 3

The Gauss-Weingarten equations in Minkowski space

As we are isometrically mapping a two dimensional manifold into Minkowski three space as opposed to Euclidean three space we find that there are some subtle changes in the preliminary stages

Let M denote a simply connected oriented 2 dimensional manifold and let $f \quad M \to \mathbb{R}^3$ be a space-like immersion, that is for all $p \in M$ and $X_p \in T_pM$, $ker(f_*)_p = \{0\}$ and $(f_*)_pX$ is space-like (where subscript $*$ means the derivative) Thus f induces a Riemannian metric g on M, the pull back $g = (f)^*(\langle,\rangle_1)$, of the scalar product $\langle,\rangle_1$ in Minkowski three space, that is $g(X,Y) = \langle f_*X, f_*Y\rangle$ Hence M is a Riemannian manifold and $f \quad (M,g) \to (\mathbb{R}^3, \langle,\rangle_1)$ is an isometric immersion We note that g is positive definite since f is spacelike

There exists local coordinates (x,y) on M called isothermal coordinates which satisfy the conditions

$$g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) > 0 \quad \text{and} \quad g(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 0$$

With isothermal coordinates (x,y) defined in some neighbourhood of $p \in M$ we may write $g = e^{\phi(x,y)}(dx^2 + dy^2)$ for some positive function e^{ϕ} For each $p \in M$ there is a local mapping $\xi \quad M \to \mathbb{R}^3$, defined by

$$\xi(p) = e^{-\phi(p)} f_x(p) * f_y(p)$$

that satisfies the following equations

$$\langle f_x, \xi\rangle_1 = \langle f_y, \xi\rangle_1 = 0$$

$$\det[f_x, f_y, \xi] > 0$$

and

$$\langle \xi, \xi \rangle_1 = -1$$

where f_x, f_y represent the partial derivative of f with respect to x and y respectively As defined ξ is called the gauss map

For all smooth vector fields Y on M and $X_p \in T_pM$ we define

$$\nabla_{X_p} Y \in T_pM \quad \text{and} \quad II_p(X_p, Y) \in \mathbb{R}^3 \text{ by}$$

$$X_p(Yf) = (f_*)_p(\nabla_{X_p} Y) + II_p(X_p, Y)\xi \tag{3 1}$$

Suppose X and Y are smooth vector fields on M then

$$X_p(Yf) = (f_*)_p(\nabla_{X_p} Y) + II_p(X_p, Y)\xi$$

$$Y_p(Xf) = (f_*)_p(\nabla_{Y_p} X) + II_p(Y_p, X)\xi$$

and subtracting we find

$$f_*[X, Y] = (X_pY - Y_pX)f = (f_*)_p(\nabla_{X_p} Y - \nabla_{Y_p} X) + (II_p(X_p, Y) - II_p(Y_p, X))\xi$$

thus

$$\nabla_{X_p} Y - \nabla_{Y_p} X = [X, Y]_p$$

$$II_p(X_p, Y) = II_p(Y_p, X)$$

Accordingly we obtain a symmetric bilinear form $II_p \quad T_pM \times T_pM \to \mathbf{R} \quad (X_p, Y_p) \to II_p(X_p, Y_p)$ where $II_p(X_p, Y_p) = II_p(X_p, Y)$ for all smooth vector fields Y with value Y_p at $p \quad II_p(\,,\,)$ is called the second fundamental form on M at p

Locally we have ξ satisfying the following

$$\begin{aligned} \langle \xi, \xi \rangle_1 &= -1 \\ X_p \langle \xi, \xi \rangle_1 &= 0 \\ 2\langle X_p \xi, \xi \rangle_1 &= 0 \end{aligned}$$

thus $X_p\xi$ is orthogonal to ξ implying $X_p\xi$ is tangent to $f(M)$ at $f(p)$,so

$$\xi_*(X) = X_p\xi = (f_*)_p(A_pX_p) \qquad \text{for some } A_pX_p \in T_pM \tag{3 2}$$

Note ξ_* and f_* are linear maps and therefore A_p $T_pM \to T_pM$ $X \to A_pX$ is also linear We note the absence of the minus sign as is usual in the case of immersions into Euclidean space

For all smooth vector fields Y on M we have

$$\begin{aligned} 0 &= \langle Yf, \xi \rangle_1 \\ 0 &= X_p \langle Yf, \xi \rangle_1 \\ &= \langle X_p(Yf), \xi \rangle_1 + \langle Y_p f, X_p \xi \rangle_1 \\ &= \langle (f_*)_p(\nabla_{X_p} Y) + II_p(X_p, Y_p)\xi, \xi \rangle_1 + \langle Y_p f, (f_*)_p(A_pX_p) \rangle_1 \\ &= II_p(X_p, Y_p)\langle \xi, \xi \rangle_1 + \langle (f_*)_p Y_p, (f_*)_p(A_pX_p) \rangle_1 \\ &= -II_p(X_p, Y_p) + g_p(Y_p, A_pX_p) \end{aligned}$$

Thus we have

$$II_p(X_p, Y_p) = g_p(Y_p, A_pX_p)$$

but $II_p(,)$ is symmetric and bilinear thus $g_p(AX,Y) = g_p(X,AY)$ showing that A is symmetric with respect to g Therefore from (3 1) we have

$$X_p(Yf) = (f_*)_p(\nabla_{X_p}Y) + g_p(Y_p, A_pX_p)\xi \tag{3 3}$$

(3 3) and (3 2) are called the Gauss-Wemgarten equations

Remark:

In addition to

$$\nabla_{X_p}Y - \nabla_{Y_p}X = [X,Y]_p \tag{3 4}$$

it is easy to check that ∇ satisfies

1 $\nabla_{\alpha X_p+\beta Y_p}Z = \alpha\nabla_{X_p}Z + \beta\nabla_{Y_p}Z \quad \forall\alpha,\beta \in \mathbf{R}, \quad X_p, Y_p \in T_pM,$ and Z a smooth vector field

2 $\nabla_{X_p}(Yf) = (X_pf)Y + f(p)\nabla_{X_p}Y \quad \forall X_p \in T_pM, f \quad M \to \mathbf{R}^3_1,$ and Y a smooth vector field

3 $\nabla_{X_p}(Y+Z) = \nabla_{X_p}Y + \nabla_{X_p}Z \quad \forall X_p \in T_pM,$ and Y,Z smooth vector fields

as well as

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{3 5}$$

Therefore ∇ is the unique Levi-Civita connection determined by the Riemannian metric g

Using the Gauss-Weingarten equations we establish the following results, the proof which can be seen in Appendix A

1 $g_p(R(X_p, Y_p)Y_p, X_p) = -\det(A_p)$ where $R(X,Y)Z$ is the curvature tensor, defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Note the minus sign (i e $-\det A$) which does not appear when mapping into Euclidean space

2 $(\nabla_{X_p} A)Y = (\nabla_{Y_p} A)X$ called Codazzi's equation where

$$(\nabla_X A)Y = \nabla_X (AY) - A \nabla_X Y$$

There exists functions Γ^k_{ij} defined near $p \in M$ so that $\nabla_{(\frac{\partial}{\partial x_i})_p} \frac{\partial}{\partial x_j} \in T_pM$ can be expressed as

$$\nabla_{(\frac{\partial}{\partial x_i})_p} \frac{\partial}{\partial x_j} = \sum_{k=1}^{2} \Gamma^k_{ij}(p)(\frac{\partial}{\partial x_k})_p, \; i,j = 1,2$$

That is the Γ^k_{ij}'s are just the components of $\nabla_{(\frac{\partial}{\partial x_i})} \frac{\partial}{\partial x_j}$ relative to the coordinates $\left(\frac{\partial}{\partial x_1}\right)_p, \left(\frac{\partial}{\partial x_2}\right)_p$ for T_pM These Γ^k_{ij}'s are called the Christoffel symbols

In the case when M has locally defined isothermal coordinates x_1, x_2 i e

$$g(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}) = g(\tfrac{\partial}{\partial x_2}, \tfrac{\partial}{\partial x_2}) = e^{\phi(x,y)}$$

$$g(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}) = 0$$

Then

$$\Gamma^1_{11} = \tfrac{1}{2}\phi_x, \quad \Gamma^1_{22} = -\tfrac{1}{2}\phi_x, \; \Gamma^1_{12} = \tfrac{1}{2}\phi_y, \; \Gamma^1_{21} = \tfrac{1}{2}\phi_y$$

$$\Gamma^2_{11} = -\tfrac{1}{2}\phi_y, \; \Gamma^2_{22} = \tfrac{1}{2}\phi_y, \quad \Gamma^2_{12} = \tfrac{1}{2}\phi_x, \; \Gamma^2_{21} = \tfrac{1}{2}\phi_x$$

Proof : see Appendix B

Chapter 4

Isometric deformations and the drehriss

Let M be defined as in the chapter 3 and let f^t be a one parameter group of isometric immersions mapping M into $\mathbb{R}^3$ Then for each t, we may assume that f^t induces the same Riemannian metric g on M, i e $g = (f^t)^*(\langle , \rangle_1)$ is independent of t Hence for each t, $f^t \quad (M, g^t) \to (\mathbb{R}^3, \langle , \rangle_1)$ is an isometric immersion

We now follow the approach of Burns and Clancy [1] with the appropriate changes Fix a $p \in M$ and consider the moving frame

$$\Omega_p^t = [f_x^t(p), f_y^t(p), \xi^t(p)]$$

along the curve $t \mapsto f^t(p) \in \mathbb{R}^3$ where

$$\xi^t(p) = e^{-\phi} f_x^t(p) * f_y^t(p)$$

For each t, f^t is an *isometric immersion* so there exists $\Theta_p^t \in SO(2,1)$, depending smoothly on t such that $\Omega_p^t = \Theta_p^t\, \Omega_p^0$ and, therefore, taking the t-derivative with $(\;= d/dt)\,,(\Theta_p^t)^T$ denoting the transpose of Θ_p^t and $(\Theta_p^t)^\dagger$ as defined in the preliminaries we have using lemma 2 7 that

$$\begin{aligned} \Omega_p^t \quad &= \quad \Theta_p^t\, \Omega_p^0 \quad = \quad \Theta_p^t\, [((\Theta_p^t)^{\dagger\dagger})^T \Theta_p^t]\, \Omega_p^0 \\ &= \quad [\Theta_p^t\, ((\Theta_p^t)^{\dagger\dagger})^T]\, \Omega_p^t \quad = \quad (\mathrm{ad}\eta^t(p))\, \Omega_p^t \end{aligned}$$

for some uniquely determined $\eta^t(p) \in \mathbb{R}^3$ So for each "*time*" t we obtain a map

$$M \to \mathbb{R}^3 \quad p \mapsto \eta^t(p)$$

called the drehriss of the deformation f^t at time t

Now if we express the formula $\Omega_p^t = (\mathrm{ad}\eta^t(p))\,\Omega_p^t$ in component form we obtain the fundamental equations

$$\begin{aligned} f_x^t(p) &= \eta^t(p) * f_x^t(p) \\ f_y^t(p) &= \eta^t(p) * f_y^t(p) \\ \xi^t(p) &= \eta^t(p) * \xi^t(p) \end{aligned} \qquad (4\ 1)$$

Furthermore, since $f_{xy} = f_{yx}$ it follows from equations (4 1) that

$$\eta_x * f_y = \eta_y * f_x$$

and consequently

$$\begin{aligned} \langle \eta_y, \xi \rangle_1 \, e^{\phi} &= \langle \eta_y, f_x * f_y \rangle_1 &= \langle \eta_y * f_x, f_y \rangle_1 \\ &= \langle \eta_x * f_y, f_y \rangle_1 &= 0 \end{aligned}$$

Thus, with a similar argument applied to $\langle \eta_x, \xi \rangle_1$, we obtain

$$\langle \eta_x, \xi \rangle_1 = \langle \eta_y, \xi \rangle_1 = 0 \qquad (4\ 2)$$

Proposition 4.1

If J is the complex structure on M which is compatible with the metric g and the orientation, then

$$\eta_* = -f_* \circ J \circ A$$

Proof:

If we put $(x^1, x^2) = (x, y), f_1 = f_x, f_2 = f_y$ and we use the summation convention, then at p we have

$$\begin{aligned}
\xi_j &= \frac{\partial}{\partial t}\frac{\partial \xi}{\partial x^j} \\
&= \frac{\partial}{\partial t} f_*(A \frac{\partial}{\partial x^j}) \\
&= \frac{\partial}{\partial t}(a^i_j)\frac{\partial f}{\partial x^i} \\
&= a^i_j \; f_i + a^i_j \; f_i \\
&= f_*(a^i_j \; \frac{\partial}{\partial x^i}) + a^i_j \; f_i
\end{aligned}$$

On the other hand, using equations (4 1) we have

$$\begin{aligned}
\xi_j &= \frac{\partial}{\partial x^j}(\eta * \xi) \\
&= \eta_j * \xi + \eta * \xi_j \\
&= -\xi * \eta_j + \eta * a^i_j f_i \\
&= -\xi * \eta_j + a^i_j \; f_i
\end{aligned}$$

Comparing these two expressions for ξ_j we obtain

$$\xi * \eta_j = -f_*(A \; \frac{\partial}{\partial x^j})$$

and, hence

$$\xi * (\xi * \eta_j) = -\xi * f_*(A \; \frac{\partial}{\partial x^j})$$

Recall from the preliminaries that $\xi * f_x = -f_y$ and $\xi * f_y = f_x$ and so

$$\xi * (\xi * \eta_j) = f_*(J \; A \; \frac{\partial}{\partial x^j})$$

since f is an isometric immersion preserving the orientation It now follows from lemma 2 5 that

$$\langle \xi, \xi \rangle_1 \eta_j - \langle \xi, \eta_j \rangle_1 \xi = f_*(J\ A\ \frac{\partial}{\partial x^j})$$

so using equation (4 2) and the fact that $\langle \xi, \xi \rangle_1 = -1$ we obtain

$$\eta_* \frac{\partial}{\partial x^j} = -(f_* \circ J \circ A) \frac{\partial}{\partial x^j}, \quad j = 1, 2$$

which proves the proposition

Chapter 5

Deformations Preserving Mean Curvature

In this section we list some of the results found by Burns and Clancy [1] and include them for completeness only Suppose that $f^t \; M \to \mathbb{R}^3$ is an isometric deformation which (as t varies) preserves mean curvature at $p \in M$ The eigenvalues $\lambda_1(p) \leq \lambda_2(p)$ of A^t the second fundamental form, are just the roots of the equation

$$\lambda^2 - 2H(p)\lambda - K(p) = 0$$

where $H(p) = (1/2)TrA^t_p$ is the mean curvature of f^t at p and $K(p)$ is the gauss curvature at p These eigenvalues are independent of t and therefore, when p is not an umbilic (i e $\lambda_1(p) \neq \lambda_2(p)$), there exists a unique $\Theta(p,t) \in [0, 2\pi)$ such that

$$A^t_p = e^{\Theta(p,t)J} A^0_p e^{-\Theta(p,t)J}$$

where J is as described in proposition 4 1 Also, when p is an umbilic we can choose $\Theta(p,t)$ arbitrarily Furthermore, since A^0_p is symmetric and J is skew-symmetric with respect to g

$$(A^t - HI)_p = e^{2\Theta(p,t)J}(A^0 - HI)_p \tag{5 1}$$

Proposition 5 1

Let $f^0 \; M \to (\mathbb{R}^3, \langle , \rangle_1)$ be an isometric immersion having constant mean curvature H If $f^t \; (M,g) \to \mathbb{R}^3$ is an isometric deformation of f^0 which preserves this constant mean curvature, then there exists a smooth function $t \mapsto \psi(t)$, depending on t only, such that

$$(A^t - HI)_p = e^{\psi(t)J}(A^0 - HI)_p$$

for every $p \in M$ and the drehriss for this deformation is given by

$$\eta^t(p) = \psi(t)[\xi^t(p) - Hf^t(p)] + \mathbf{E}^t$$

where $\mathbf{E}^t \in \mathbb{R}^3$ does not depend on p

Proof :

Given that each immersion f^t is of constant mean curvature H, i e $TrA^t = 2H$ for all t and $g = e^{\phi(x,y)}(dx^2 + dy^2)$, we may write A^t in terms of local isothermal coordinates (x, y) on M as follows

$$A^t = \begin{pmatrix} a^t_{11} & a^t_{12} \\ a^t_{12} & a^t_{22} \end{pmatrix}$$

It can now be shown that Codazzi's equation for A^t is equivalent to the Cauchy-Riemann equations for the complex function

$$\Psi^t(x + iy) = ((a^t_{11} - a^t_{22}) - 2ia^t_{12})e^{\phi}$$

This was first observed by H Hopf and a proof can be seen in Appendix C We note that the umbilics of the immersion f^t (which are the zeros of Ψ^t) are isolated Note, we rule out the case $\Psi^t \equiv 0$ since this corresponds to the immersed surface being a portion of the standard hyperbolic sphere in $\mathbb{R}^3$ Therefore, the function $\Theta(p, t)$ in equation (5 1) is a smooth function defined for all $p \in M$ except for these isolated umbilics If we fix t and recall that $\nabla J = 0$, then by applying Codazzi's equation to both sides of equation (5 1) we find that since

$$(\nabla_{\frac{\partial}{\partial x}}(A^t - HI)_p)\frac{\partial}{\partial y} = (\nabla_{\frac{\partial}{\partial y}}(A^t - HI)_p)\frac{\partial}{\partial x}$$

we must have

$$(\nabla_{\frac{\partial}{\partial x}} e^{2\Theta(p,t)J}(A^0 - HI)_p)\frac{\partial}{\partial y} = (\nabla_{\frac{\partial}{\partial y}} e^{2\Theta(p,t)J}(A^0 - HI)_p)\frac{\partial}{\partial x}$$

and so

$$\nabla_{\frac{\partial}{\partial x}}\left(e^{2\Theta(p,t)J}(A^0 - HI)_p\frac{\partial}{\partial y}\right) - e^{2\Theta(p,t)J}(A^0 - HI)_p\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} =$$

$$\nabla_{\frac{\partial}{\partial y}}\left(e^{2\Theta(p,t)J}(A^0 - HI)_p\frac{\partial}{\partial x}\right) - e^{2\Theta(p,t)J}(A^0 - HI)_p\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x}$$

implying

$$2J\Theta_x e^{2\Theta(p,t)J}(A^0 - HI)_p\frac{\partial}{\partial y} + e^{2\Theta(p,t)J}\nabla_{\frac{\partial}{\partial x}}(A^0 - HI)_p\frac{\partial}{\partial y} =$$

$$2J\Theta_y e^{2\Theta(p,t)J}(A^0 - HI)_p\frac{\partial}{\partial x} + e^{2\Theta(p,t)J}\nabla_{\frac{\partial}{\partial y}}(A^0 - HI)_p\frac{\partial}{\partial x}$$

Hence $\Theta_x = \Theta_y = 0$ i e Θ is independent of p Accordingly, the first statement follows if we put $\psi(t) = 2\Theta(p,t)$ For the second statement we take the t-derivative (with p fixed) across the equation

$$(A^t - HI)_p = e^{\psi(t)J}(A^0 - HI)_p$$

to obtain

$$A^t_p = \psi(t)\ Je^{\psi(t)J}(A^0 - HI)_p = \psi(t)\ J(A^t - HI)_p$$

and from proposition 4 1 we have

$$\begin{aligned} \eta_* &= -f_* \circ J\ A &&= -f_* \circ J(\psi\ J(A - HI)) \\ &= \psi\ f_* \circ (A - HI) &&= \psi\ (f_* \circ A - Hf_*) \\ &= \psi\ (\xi_* - Hf_*) &&= (\psi\ (\xi - Hf))_* \end{aligned}$$

from which the second statement follows

Chapter 6

The manifold

Throughout this section we will assume that the simply-connected Riemann Surface M with metric g admits a 1-parameter group of isometries

A one parameter group of isometries of M is a family $\{\Phi_t \quad M \to M | t \in \mathbb{R}\}$ of mappings with the following three properties

a $\{\Phi_t \quad M \to M | t \in \mathbb{R}\}$ is a group under composition i e

$$\Phi_t \circ \Phi_s = \Phi_{t+s} \quad \text{for all } s, t \in \mathbb{R} \text{ and } \Phi_0 = id$$

b For all $p \in M$ the mapping $t \mapsto \Phi_t(p)$ is differentiable

c For all $t \in \mathbb{R}, \Phi_t \quad M \to M$ is an isometry

The uniformization theorem states that every simply connected Riemann surface is conformally equivalent to just one of

i The Riemann Sphere S^2
ii The Complex Plane $\mathbf{C}$
iii The Open Unit Disk $\mathcal{U}$

hence there exists a conformal diffeomorphism $F \quad \Delta \to M$ where $\Delta = S^2, \mathbf{C}$, or $\mathcal{U}$, each equipped with its standard metric

Let $\tilde{g} = F^*(g)$ be the metric induced on Δ by F Then $F \quad (\Delta, \tilde{g}) \to (M, g)$ is isometric The one parameter group $\{\Phi_t \quad M \to M | t \in \mathbb{R}\}$ of isometries of M induces a one parameter group $\{\psi_t \quad \Delta \to \Delta | t \in \mathbb{R}\}$ of isometries of Δ Hence we let Δ equal $S^2, \mathbf{C}$ and $\mathcal{U}$ and investigate all one parameter groups of isometries

If $\Delta = S^2$ then f immerses M into a piece of the hyperbolic 2-sphere, a proof of this is found in Appendix E If $\Delta \neq S^2$ then we may assume (see smyth [6] for example) that after a conformal change in the model, the group ψ_t is one of the following

(a1) all rotations about a fixed point of $\mathcal{U}$ (the origin 0, say)

(a2) all automorphisms of $\mathcal{U}$ fixing one boundary point ($z = 1$, say)

(a3) all automorphisms of $\mathcal{U}$ fixing two boundary points ($z = \pm 1$, say)

(b1) all translations in a fixed direction of $\mathbf{C}$ (the y-axis, say)

(b2) all rotations about a fixed point of $\mathbf{C}$ (the origin 0,say)

Now under the covering map $z \rightarrow e^z$, the regions in (a1) and (b2) correspond to the half-plane $Re(z) < 0$ and $\mathbf{C}$, respectively, and the group of translations parallel to the y-axis If we transform the disk $\mathcal{U}$ into the half plane $Re(z) < 0$ so that 1 is transformed to ∞ then the group in (a2) must transform into the group of translations parallel to the y-axis If we transform the disk $\mathcal{U}$ into the strip $a < Re(z) < b$ so that ± 1 are transformed to $y = \pm\infty$ then the group in (a3) must transform into the group of translations parallel to the y-axis

Let $V = \{z \in \mathbf{C} \,|\sigma < Re(z) < \tau\}$ Then for each of the regions $\mathcal{R}$ occurmg in the cases above there is a conformal mapping $\pi \;\; V \rightarrow \mathcal{R}$ such that ψ_t pulls back under π to the group of vertical translations of V

The relevant information for the five cases is as follows

(a1) $(\sigma, \tau) = (-\infty, 1)$, π is a covering map onto $D - \{0\}$ and $\pi\{x = -\infty\} = 0$

(a2) $(\sigma, \tau) = (-\infty, 1)$, π is a diffeomorphism

(a3) $(\sigma, \tau) = (a, b)$, π is a diffeomorphism and a, b are any fixed constants of our choosing

(b1) $(\sigma, \tau) = (-\infty, \infty)$, π is a diffeomorphism

(b2) $(\sigma, \tau) = (-\infty, \infty)$, π is a covering map onto $\mathbf{C} - \{0\}$ and $\pi\{x = -\infty\} = 0$

The quantities arising on V from the induced immersion $f \circ \pi$ are denoted by the same letters as before Thus $g = e^{\phi}|dz|^2$, the function ϕ depends only on x and we assume $M = \{z \in \mathbf{C} \,|\sigma < Re(z) < \tau\}$ with ψ_t the group of vertical translations of M

Chapter 7

CMC-Surfaces with Internal Symmetry

Throughout this section we will assume that M is a simply-connected surface with Riemannian metric g which admits a 1-parameter group of isometries

$$\psi^t \quad (M,g) \to (M,g)$$

We wish to classify all isometric immersions

$$f \quad (M,g) \to (\mathbb{R}^3, \langle , \rangle_1)$$

which have constant mean curvature To this end observe that the 1-parameter family of immersions

$$f^t \quad (M,g) \to \mathbb{R}^3 \quad p \mapsto f(\psi^t(p))$$

is an isometric deformation of $f = f^0$ which preserves the constant mean curvature H Now, since we may assume that conformally

$$(M,g) = (\ \{(x,y) \in \mathbf{R}^2 \quad a < x < b\} \quad \text{for some } a, b \in \mathbf{R}, e^{\phi(x)}(dx^2 + dy^2)\ )$$

and the isometry $\psi^t(x,y) = (x, y+t)$ for all t therefore for all x, y and t we have

$$\begin{aligned} f^t(x,y) &= f^0(x, y+t) \\ \xi^t(x,y) &= \xi^0(x, y+t) \\ A^t(x,y) &= A^0(x, y+t) \end{aligned}$$

In particular, $A^t(x,0) = A^0(x,t)\ \ \forall(x,t)$ and combining this with proposition 5 1 we obtain

$$(A^0 - HI)_{(x,t)} = (A^t - HI)_{(x,0)} = e^{\psi(t)J}(A^0 - HI)_{(x,0)}$$

If we replace t by y, then

$$(A^0 - HI)_{(x,y)} = e^{\psi(y)J}(A^0 - HI)_{(x,0)}$$

and applying Codazzi's equation we find

$$\psi(y) = -\alpha y + \beta \text{ for some constants } \alpha, \beta \in \mathbf{R}$$

Thus, $\psi\,(t) \equiv -\alpha$ and, from proposition 5 1, we see that the drehriss of this deformation is

$$\eta^t(x,y) = -\alpha\xi^t(x,y) + (\mathbf{E}^t + \alpha H f^t(x,y))$$

For later use we also observe that $\beta = \psi(0) = 2\pi n$ for some integer n, which we may choose to be zero Therefore, the constant α is uniquely determined by the condition

$$(A^0 - HI)_{(x,y)} = e^{-\alpha y J}(A^0 - HI)_{(x,0)} \tag{7 1}$$

Now, from the identity $f^t(x,y) \equiv f^0(x, y+t)$ it follows that

$$\overset{t}{f}\;(x,y) = f^0_2(x, y+t)$$

$$\overset{t}{f_1}\;(x,y) = f^0_{21}(x, y+t)$$

$$\overset{t}{f_2}\;(x,y) = f^0_{22}(x, y+t)$$

$$\overset{t}{\xi}\;(x,y) = \xi^0_2(x, y+t)$$

where the subscripts 1 and 2 denote partial derivatives with respect to the first and second variables, respectively

From now on we set $t = 0$ and drop the superscript "0" from all functions so that using the fundamental equations (4 1) we obtain

$$\begin{aligned} f_{xy}(x,y) &= f_x\,(x,y) &= \eta(x,y) * f_x(x,y) \\ f_{yy}(x,y) &= f_y\,(x,y) &= \eta(x,y) * f_y(x,y) \\ \xi_y(x,y) &= \xi\,(x,y) &= \eta(x,y) * \xi(x,y) \end{aligned}$$

where $\eta(x,y) = -\alpha\xi(x,y) + (\mathbf{E} + \alpha H f(x,y))$ for some constant vector $\mathbf{E} \in \mathbb{R}^3$ Therefore,

$$\begin{aligned} f_{xy} &= -\alpha\xi * f_x + (\mathbf{E} + \alpha H f) * f_x \\ f_{yy} &= -\alpha\xi * f_y + (\mathbf{E} + \alpha H f) * f_y \\ \xi_y &= (\mathbf{E} + \alpha H f) * \xi \end{aligned} \tag{7 2}$$

However, the coordinate system (x, y) is isothermal and ξ was defined so that $\xi * f_x = -f_y$ and $\xi * f_y = f_x$ Therefore, we obtain the equations

$$f_{xy} = \alpha f_y + (\mathbf{E} + \alpha H f) * f_x \tag{7 3}$$

$$f_{yy} = -\alpha f_x + (\mathbf{E} + \alpha H f) * f_y \tag{7 4}$$

When $\alpha \neq 0$ one obtains from the integrability conditions, $f_{xyy} = f_{yyx}$ and $f_{xy} = f_{yx}$, the additional equation

$$f_{xx} + f_{yy} = 2H f_x * f_y \tag{7 5}$$

furthermore when $\alpha \neq 0$ and $H \neq 0$ equations (7 3), (7 4) and (7 5) can be reduced to

$$f_{xy} = \alpha f_y + \alpha H f * f_x \tag{7 6}$$

$$f_{yy} = -\alpha f_x + \alpha H f * f_y \tag{7 7}$$

$$f_{xx} + f_{yy} = 2H f_x * f_y \tag{7 8}$$

by replacing $f(x,y)$ with $f(x,y) - \frac{1}{\alpha H}\mathbf{E}$, i e a simple translation and we note for future reference that (7 2) simplifies to

$$\xi_y = \alpha H f * \xi \tag{7 9}$$

Hence the above transformation allows us to assume that $\mathbf{E} = (0,0,0)^T$ in the original equations The differential equations (7 6),(7 7) and (7 8) can be further simplified by replacing $f(x,y)$ by $Hf(x,y)$ to give

$$f_{xy} = \alpha f_y + \alpha f * f_x \tag{7 10}$$

$$f_{yy} = -\alpha f_x + \alpha f * f_y \tag{7 11}$$

$$f_{xx} + f_{yy} = 2 f_x * f_y \tag{7 12}$$

i e we may further assume that $H = 1$

When $\alpha = 0$ equations (7 3), (7 4) and (7 5)reduce to the form

$$\begin{aligned} f_{xy} &= \mathbf{E} * f_x \\ f_{yy} &= \mathbf{E} * f_y \\ \xi_y &= \mathbf{E} * \xi \end{aligned}$$

We now determine a necessary and sufficient condition in terms of isothermal coordinates (x, y) for a spacelike immersion f into Minkowski three space to be of constant mean curvature H As

$$\langle f_x, f_x \rangle_1 = \langle f_y, f_y \rangle_1 > 0 \tag{7 13}$$

$$\langle f_x, f_y \rangle_1 = 0 \tag{7 14}$$

we conclude from (7 13) that

$$\langle f_{xx}, f_x \rangle_1 = \langle f_{xy}, f_y \rangle_1 \tag{7 15}$$

and by (7 14)

$$\langle f_{xy}, f_y \rangle_1 + \langle f_x, f_{yy} \rangle_1 = 0 \tag{7 16}$$

Using (7 15) and (7 16) we see that

$$\langle f_{xx} + f_{yy}, f_x \rangle_1 = 0 \tag{7 17}$$

and similarly one can show that

$$\langle f_{xx} + f_{yy}, f_y \rangle_1 = 0 \tag{7 18}$$

implying that $f_{xx} + f_{yy} = \vartheta(x, y)\xi$ for some function $\vartheta(x, y)$

Now

$$
\begin{aligned}
\langle f_{xx} + f_{yy}, \xi \rangle_1 &= \langle f_{xx}, \xi \rangle_1 + \langle f_{yy}, \xi \rangle_1 \\
&= (\langle f_x, \xi \rangle_1)_x - \langle f_x, \xi_x \rangle_1 + (\langle f_y, \xi \rangle_1)_y - \langle f_y, \xi_y \rangle_1 \\
&= -\langle f_x, a_{11} f_x + a_{12} f_y \rangle_1 - \langle f_y, a_{21} f_x + a_{22} f_y \rangle_1 \\
&= -2He^{\phi}
\end{aligned}
$$

and since $\langle \vartheta(x,y)\xi, \xi \rangle_1 = -\vartheta(x,y)$ we have that

$$f_{xx} + f_{yy} = 2H f_x * f_y$$

That is the integrability condition (7 5) (in isothermal coordinates) is a necessary and sufficient condition for the immersion to be of constant mean curvature and so it holds whether $\alpha = 0$ or not

Chapter 8

The Associates

With the notation of the previous sections, let $f \quad (M,g) \to \mathbb{R}^3$ be an isometric immersion of constant mean curvature H If A^0 denotes the second fundamental form of f, then, by definition the associates of f are the isometric immersions in the 1-parameter family $f^t \quad (M,g) \to \mathbb{R}^3$ which have their second fundamental forms A^t determined by

$$(A^t - HI)_p = e^{tJ}(A^0 - HI)_p \quad \forall\, p \in M \tag{8 1}$$

These immersions have the same constant mean curvature as f If we begin with one of the immersions f as determined by in chapter 7, that is, the constants $H, e^{\phi(x_0)}, \alpha$ and $\mathbf{E}$ are specified, then the question arises how should these parameters be varied to obtain the associates of f? At once we see that H remains constant and so also does $\lambda(x_0)$ since it determines the metric g at (x_0, y_0) which is the same for all associates Now, let $\alpha(t)$ and $\mathbf{E}^t$ denote the remaining parameters which correspond to the associate f^t From the previous section, see equation (7 1), $\alpha(t)$ is uniquely determined by the following

$$\begin{aligned} e^{\alpha(t)yJ}(A^t - HI)_{(x,0)} &= (A^t - HI)_{(x,y)} \\ &= e^{tJ}(A^0 - HI)_{(x,y)} \\ &= e^{tJ}e^{\alpha(0)yJ}(A^0 - HI)_{(x,0)} \\ &= e^{\alpha(0)yJ}e^{tJ}(A^0 - HI)_{(x,0)} \\ &= e^{\alpha(0)yJ}(A^t - HI)_{(x,0)} \end{aligned}$$

Therefore, $e^{\alpha(t)yJ} \equiv e^{\alpha(0)yJ}$ and $\alpha(t) = \alpha(0) + 2n\pi$ where n is some integer

Remark.

If as in the previous section $\psi^s(x,y) = (x, y+s) \quad s \in \mathbb{R}$ denotes the 1-parameter group of internal isometries, then $f \circ \psi^s$ has second fundamental form $e^{\alpha sJ}(A^0 - HI) + HI$ while the associate f^t has second fundamental form $e^{tJ}(A^0 - HI) + HI$ Therefore up to a Hyperbolic motion we have

$$f^t = f \circ \psi^{t/\alpha} \quad \forall \alpha \neq 0$$

That is when $\alpha \neq 0$, the associates no longer generate "new surfaces" but rather (up to a Hyperbolic motion) correspond to the flow of the internal symmetry along f

Chapter 9

The second fundamental form

Lemma 9 1

When $\alpha = 0$ we have

$$\langle \mathbf{E}, f \rangle_1 = c_1 y + c_2 x + c_3 + H \int e^{\phi} dx$$

for some $c_1, c_2, c_3 \in \mathbb{R}$ and the second fundamental form, A, satisfies

$$A - HI = e^{-\phi} \begin{pmatrix} c_2 & c_1 \\ c_1 & -c_2 \end{pmatrix} \tag{9 1}$$

Proof

When $\alpha = 0$ we have that every constant mean curvature surface which has continuous internal symmetry satisfies the following differential equations

$$f_{xy} = \mathbf{E} * f_x \tag{9 2}$$

$$f_{yy} = \mathbf{E} * f_y \tag{9 3}$$

$$\xi_y = \mathbf{E} * \xi \tag{9 4}$$

$$f_{xx} + f_{yy} = 2H f_x * f_y \tag{9 5}$$

and hence

$$\langle E, f_{xy} \rangle_1 = \langle E, f_{yy} \rangle_1 = 0$$

which implies

$$\langle E, f_y \rangle_1 = c_1$$

$$\begin{aligned}
\langle E, f_{xx}\rangle_1 &= \langle E, f_{xx}\rangle_1 + \langle E, f_{yy}\rangle_1 \\
&= 2H\langle E, f_x * f_y\rangle_1 \\
&= 2H\langle E * f_x, f_y\rangle_1 \\
&= 2H\langle f_{xy}, f_y\rangle_1 \\
&= H(\langle f_y, f_y\rangle_1)_x \\
&= H(e^{\phi})_x \\
&= (He^{\phi})_x
\end{aligned}$$

Hence

$$\langle E, f_x\rangle_1 = He^{\phi} + c_2(y)$$

but

$$\langle E, f_{xy}\rangle_1 = 0$$

and so

$$\langle E, f_x\rangle_1 = He^{\phi} + c_2$$

$$\xi_y = E * \xi$$

$$\frac{\partial}{\partial y}\xi = e^{-\phi} E * (f_x * f_y)$$

$$f_*\left(A\frac{\partial}{\partial y}\right) = e^{-\phi}(\langle E, f_x\rangle_1 f_y - \langle E, f_y\rangle_1 f_x)$$

$$f_*\left(a_{21}\frac{\partial}{\partial x} + a_{22}\frac{\partial}{\partial y}\right) = e^{-\phi}((He^{\phi} + c_2)f_y - c_1 f_x)$$

$$a_{21}f_x + a_{22}f_y = (H + c_2e^{-\phi})f_y - c_1e^{-\phi}f_x$$

and hence

$$a_{22} = H + c_2 e^{-\phi} \tag{9 6}$$

$$a_{21} = -c_1 e^{-\phi} \tag{9 7}$$

we recall that $a_{12} = a_{21}$ and $a_{11} + a_{22} = 2H$ and so

$$A = HI + e^{-\phi} \begin{pmatrix} c_2 & c_1 \\ c_1 & -c_2 \end{pmatrix} \tag{9 8}$$

this completes the proof Following the approach of Burns and Clancy [1] we also have the following lemma

Lemma 9.2

When $\alpha \neq 0$

$$\langle \mathbf{E} + \frac{\alpha H f}{2}, f \rangle_1 = a\cos(\alpha y + t)e^{\alpha x} + H \int e^{\phi} dx$$

for some $a \in \mathbb{R}$ and the second fundamental form, A, satisfies

$$A - HI = a\alpha e^{-\phi + \alpha x} \begin{pmatrix} \cos(\alpha y + t) & \sin(\alpha y + t) \\ \sin(\alpha y + t) & -\cos(\alpha y + t) \end{pmatrix}$$

Proof.

We have

$$\langle \mathbf{E} + \alpha H f, f_{xy} \rangle_1 = (\langle \mathbf{E} + \alpha H f, f_y \rangle_1)_x = (\langle \mathbf{E} + \frac{\alpha H f}{2}, f \rangle_1)_{xy} \tag{9 9}$$

$$\begin{aligned}\langle \mathbf{E} + \alpha H f, f_{yy}\rangle_1 &= (\langle \mathbf{E} + \alpha H f, f_y\rangle_1)_y - \alpha H \langle f_y, f_y\rangle_1 \\ &= (\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_{yy} - \alpha H e^{\phi} \end{aligned} \tag{9 10}$$

$$\begin{aligned}\langle \mathbf{E} + \alpha H f, f_{xx}\rangle_1 &= (\langle \mathbf{E} + \alpha H f, f_x\rangle_1)_x - \alpha H \langle f_x, f_x\rangle_1 \\ &= (\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_{xx} - \alpha H e^{\phi} \end{aligned} \tag{9 11}$$

Using (9 9) we have

$$\begin{aligned}(\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_{xy} &= \langle \mathbf{E} + \alpha H f, f_{xy}\rangle_1 \\ &= \langle \mathbf{E} + \alpha H f, \alpha f_y + (\mathbf{E} + \alpha H f) * f_x\rangle_1 \quad \text{by (7 3)} \\ &= \alpha \langle \mathbf{E} + \alpha H f, f_y\rangle_1 \\ &= \alpha(\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_y \end{aligned} \tag{9 12}$$

so that

$$(\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_y = \Gamma(y) e^{\alpha x} \tag{9 13}$$

for some function $\Gamma(y)$ Differentiating both sides of this equation by y we have that

$$\begin{aligned}\Gamma''(y) e^{\alpha x} &= (\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_{yyy} \\ &= (\langle \mathbf{E} + \alpha H f, f_{yy}\rangle_1 + \alpha H e^{\phi})_y \quad \text{by (9 10)} \\ &= (\langle \mathbf{E} + \alpha H f, f_{yy}\rangle_1)_y \\ &= (\langle \mathbf{E} + \alpha H f, -f_{xx} + 2H e^{\phi} \xi\rangle_1)_y \quad \text{by (7 5)} \end{aligned}$$

$$= -((\langle \mathbf{E} + \alpha H f, f_{xx} \rangle_1)_y + 2He^{\phi}((\langle \mathbf{E} + \alpha H f, \xi \rangle_1)_y$$

$$= -((\langle \mathbf{E} + \alpha H f, f_{xx} \rangle_1)_y + 2He^{\phi}(\alpha H \langle f_y, \xi \rangle_1 + \langle \mathbf{E} + \alpha H f, \xi_y \rangle_1)$$

$$= -((\langle \mathbf{E} + \alpha H f, f_{xx} \rangle_1)_y \quad \text{by (7 2)}$$

$$= -((\langle \mathbf{E} + \frac{\alpha H f}{2}, f \rangle_1)_{xxy}$$

$$= -(\Gamma(y) e^{\alpha x})_{xx} \text{ by (9 13)}$$

$$= -\Gamma(y)\alpha^2 e^{\alpha x}$$

Hence we have

$$\begin{aligned}
\Gamma''(y) &= -\alpha^2 \Gamma(y) \\
\Gamma(y) &= -a\alpha \sin(\alpha y + t) \text{ for some } a, t \in \mathbb{R} \\
((\mathbf{E} + \frac{\alpha H f}{2}, f \rangle_1)_y &= -a\alpha \sin(\alpha y + t) e^{\alpha x} \\
\langle \mathbf{E} + \frac{\alpha H f}{2}, f \rangle_1 &= a \cos(\alpha y + t) e^{\alpha x} + \chi(x)
\end{aligned}$$

Also

$$\begin{aligned}
\chi'(x) + a\alpha\cos(\alpha y + t)e^{\alpha x} &= (\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_x \\
&= \langle \mathbf{E} + \alpha H f, f_x\rangle_1 \\
&= \langle \mathbf{E} + \alpha H f, -\frac{1}{\alpha} f_{yy} + \frac{1}{\alpha}(\mathbf{E} + \alpha H f) * f_y\rangle_1 \quad \text{by (7 4)} \\
&= -\frac{1}{\alpha}\langle \mathbf{E} + \alpha H f, f_{yy}\rangle_1 \\
&= -\frac{1}{\alpha}((\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_{yy} - \alpha H e^{\phi}) \quad \text{by (9 10)} \\
&= -\frac{1}{\alpha}(-a\alpha^2\cos(\alpha y + t)e^{\alpha x} - \alpha H e^{\phi}) \\
\chi'(x) &= He^{\phi}
\end{aligned}$$

$$\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1 = a\cos(\alpha y + t)e^{\alpha x} + H\int e^{\phi}dx$$

proving the first part

Now

$$\begin{aligned}
\xi_y &= (\mathbf{E} + \alpha H f) * \xi \\
f_*(A\frac{\partial}{\partial y}) &= e^{-\phi}(\mathbf{E} + \alpha H f) * (f_x * f_y) \\
f_*(a_{21}\frac{\partial}{\partial x} + a_{22}\frac{\partial}{\partial y}) &= e^{-\phi}(\langle \mathbf{E} + \alpha H f, f_x\rangle_1 f_y - \langle \mathbf{E} + \alpha H f, f_y\rangle_1 f_x) \\
a_{21}\frac{\partial f}{\partial x} + a_{22}\frac{\partial f}{\partial y} &= e^{-\phi}(((\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_x f_y - ((\langle \mathbf{E} + \frac{\alpha H f}{2}, f\rangle_1)_y f_x) \\
a_{21} f_x + a_{22} f_y &= e^{-\phi}((a\alpha\cos(\alpha y + t)e^{\alpha x} + He^{\phi})f_y + a\alpha\sin(\alpha y + t)e^{\alpha x} f_x)
\end{aligned}$$

Hence

$$\begin{aligned}
a_{12} = a_{21} &= a\alpha e^{-\phi+\alpha x}\sin(\alpha y + t) \\
a_{22} &= H + a\alpha e^{-\phi+\alpha x}\cos(\alpha y + t) \\
a_{11} = 2H - a_{22} &= H - a\alpha e^{-\phi+\alpha x}\cos(\alpha y + t) \\
A - HI &= a\alpha e^{-\phi+\alpha x}\begin{pmatrix} -\cos(\alpha y + t) & \sin(\alpha y + t) \\ \sin(\alpha y + t) & \cos(\alpha y + t) \end{pmatrix}
\end{aligned}$$

Thus proving the lemma

Lemma 9 3

For every $\alpha \in \mathbb{R}$ we may assume without loss of generality that the second fundamental form A may be written as

$$A = HI - ce^{-\phi+\alpha x}\begin{pmatrix} -\cos\alpha y & \sin\alpha y \\ \sin\alpha y & \cos\alpha y \end{pmatrix} \tag{9 14}$$

where $c \in \mathbb{R}, c > 0$

Proof

We recall when $\alpha \neq 0$ that the associates, f_θ, of the immersion f do not generate new surfaces and that the corresponding second fundamental form A_θ is given by

$$A_\theta = e^{\theta J}(A - HI) + HI$$

When $\alpha \neq 0$ we then have

$$A_\theta - HI = a\alpha e^{-\phi+\alpha x}\begin{pmatrix} -\cos(\alpha y + t - \theta) & \sin(\alpha y + t - \theta) \\ \sin(\alpha y + t - \theta) & \cos(\alpha y + t - \theta) \end{pmatrix} \tag{9 15}$$

By choosing $\theta = t$ or $\theta = t + \pi$ if necessary we have

$$A_\theta = HI - ce^{-\phi+\alpha x}\begin{pmatrix} -\cos\alpha y & \sin\alpha y \\ \sin\alpha y & \cos\alpha y \end{pmatrix} \tag{9 16}$$

where $c = |a\alpha| > 0$

When $\alpha = 0$ we have

$$A_\theta - HI = e^{-\phi}\begin{pmatrix} c_2\cos\theta - c_1\sin\theta & c_2\cos\theta + c_1\sin\theta \\ c_2\cos\theta + c_1\sin\theta & -c_2\cos\theta + c_1\sin\theta \end{pmatrix} \tag{9 17}$$

Choosing $\theta = \tan^{-1}(-\frac{c_1}{c_2})$ or $\theta = \tan^{-1}(-\frac{c_1}{c_2}) + \pi$ we may assume $c_1 = 0$ and $c_2 > 0$ giving

$$A_\theta = HI - ce^{-\phi}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9 18}$$

where $c = c_2$ Hence we may assume

$$A = HI - ce^{-\phi+\alpha x}\begin{pmatrix} -\cos\alpha y & \sin\alpha y \\ \sin\alpha y & \cos\alpha y \end{pmatrix} \tag{9 19}$$

for every $\alpha \in \mathbb{R}$ where $\mathbf{c} > 0$ This completes the proof of lemma 9 3

Lemma 9.4

When $\alpha = 0$ we may assume without loss of generality that the set of differential equations (9 2),(9 3), (9 4) and (9 5) reduce to three

$$\begin{aligned} f_y &= \mathbf{E} * f & (9\ 20) \\ \xi_y &= \mathbf{E} * \xi & (9\ 21) \\ f_{xx} + f_{yy} &= 2Hf_x * f_y & (9\ 22) \end{aligned}$$

Proof:

By (9 2) and (9 3) we have

$$\begin{aligned} f_{xy} &= \mathbf{E} * f_x & (9\ 23) \\ f_{yy} &= \mathbf{E} * f_y & (9\ 24) \end{aligned}$$

which simplify to

$$f_y = \mathbf{E} * f + v$$

for some $v \in \mathbb{R}^3$ Using lemma 9 1 and lemma 9 3 we may assume $\langle E, f_y \rangle_1 = 0$

and hence we may assume $\langle \mathbf{E}, v \rangle_1 = 0$ i e $v \in \mathbf{E}^{\perp}$ Thus $v = \mathbf{E} * b$ for some $b \in \mathbb{R}^3$ Replacing $f(x, y)$ with $f(x, y) + b$ i e a translation we find we may assume $v = (0, 0, 0)^T$ This concludes the proof

Chapter 10

Conformal transformations

With the conformal structure determined by g and the given orientation, we have that M is a Riemann surface and in terms of local conformal coordinate $z = x + iy$ we write $g = e^{\phi}|dz|^2$, where ϕ is a real function of z and $A = (a_{ij})$ with respect to the coordinate field $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$

Let $h \quad U \to h(U) \subset M$ and $\tilde{h} \quad \tilde{U} \to \tilde{h}(\tilde{U}) \subset M$ be two positively oriented conformal parameterisations of M about a point $p \in M$ with $h(U) = \tilde{h}(\tilde{U})$ So that $\vartheta \; = h^{-1} \circ \tilde{h} \quad \tilde{U} \to U$ is a bijective holomorphic mapping Also let $z = x + iy$ be the local coordinates on $h(U)$ and $w = \tilde{x} + i\tilde{y}$ be the local coordinates on $\tilde{h}(\tilde{U})$

Now

$$g = e^{\phi}|dz|^2 = e^{\phi}\left|\frac{\partial z}{\partial w}dw\right|^2 = e^{\phi}\left|\frac{\partial z}{\partial w}\right|^2 |dw|^2 = e^{\tilde{\phi}}|dw|^2$$

and hence

$$e^{\tilde{\phi}} = \left|\frac{\partial z}{\partial w}\right|^2 e^{\phi} \tag{10 1}$$

From chapter 5 we have

$$\Psi = \{(a_{11} - a_{22}) - 2ia_{12}\}e^{\phi} \tag{10 2}$$

is a holomorphic function It can easily be shown that in terms of local coordinates

$$\Psi = 4g(A\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$$

Since

$$\frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} \quad \text{and} \quad dz = \frac{\partial z}{\partial w}dw$$

we see $\Psi \; dz^2$ is a holomorphic quadratic differential independent of the coordinate system Now

$$\Psi = 4g(A\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = 4g(A\frac{\partial w}{\partial z}\frac{\partial}{\partial w}, \frac{\partial w}{\partial z}, \frac{\partial}{\partial z}) = \left|\frac{\partial w}{\partial z}\right|^2 4g(A\frac{\partial}{\partial z}, \frac{\partial}{\partial z}) = \left(\frac{\partial w}{\partial z}\right)^2 \tilde{\Psi}$$

and hence

$$\tilde{\Psi} = \left(\frac{\partial z}{\partial w}\right)^2 \Psi \tag{10 3}$$

By lemma 9 3 we may assume

$$A = \begin{pmatrix} H + ce^{-\phi+\alpha x}\cos\alpha y & -ce^{-\phi+\alpha x}\sin\alpha y \\ -ce^{-\phi+\alpha x}\sin\alpha y & H - ce^{-\phi+\alpha x}\cos\alpha y \end{pmatrix}$$

and so from (10 2) we have

$$\begin{aligned} \Psi &= \{(a_{11} - a_{22}) - 2\imath a_{12}\}e^{\phi} \\ &= 2ce^{\alpha x}\cos\alpha y + 2\imath ce^{\alpha x}\sin\alpha y \end{aligned}$$

or more simply

$$\Psi(z) = 2ce^{\alpha z} \tag{10 4}$$

this along with (10 3) gives

$$\Psi(w) = \left(\frac{\partial z(w)}{\partial w}\right)^2 2ce^{\alpha z(w)} \tag{10 5}$$

If $\alpha \neq 0$ we let $z = \frac{1}{\alpha}(w + \log\frac{\alpha^2}{c})$ then by (10 1)

$$e^{\tilde{\phi}} = |z'(w)|^2 e^{\phi(Re(z(w)))} = \frac{1}{\alpha^2}e^{\phi(\frac{1}{\alpha}(\tilde{x}+\log\frac{\alpha^2}{c}))}$$

which is still just some function of $\tilde{x}$ and using (10 5) we have

$$\tilde{\Psi}(w) = (z'(w))^2 2ce^{\alpha z(w)} = \frac{1}{\alpha^2}2ce^{w+\log\frac{\alpha^2}{c}} = 2e^w$$

thus comparing this to (10 4) we may assume that $\alpha = 1$ and $c = 1$

If $\alpha = 0$ we let $z = \frac{1}{\sqrt{c}} w$ then by (10 1)

$$e^{\tilde{\phi}} = |z'(w)|^2 e^{\phi(Re(z(w)))} = e^{\phi(\frac{1}{\sqrt{c}}\tilde{x}) - \log c)}$$

which is again just some function of $\tilde{x}$ and using (10 5) we have

$$\tilde{\Psi} = (z'(w))^2 2c = 2$$

thus comparing this to (10 4) when $\alpha = 0$ we may assume $c = 1$

Chapter 11

The metric

As shown in the preliminaries the Gauss curvature $K(p)$ is given by

$$K(p) = g_p(R(X_p, Y_p)Y_p, X_p) = -\det(A_p)$$

computing the left hand side locally with $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ we find

$$K = -\frac{1}{2}e^{-\phi}\Delta\phi$$

relative to the local coordinates (x, y) A proof of this is given in Appendix D Since ϕ depends on x only we have

$$\Delta\phi = \frac{\partial^2\phi}{\partial x^2}$$

From the previous chapter

$$A = HI - e^{-\phi+\alpha x}\begin{pmatrix} -\cos\alpha y & \sin\alpha y \\ \sin\alpha y & \cos\alpha y \end{pmatrix}$$

where $\alpha = 0$ or 1 Hence

$$\det A = H^2 - e^{-2\phi+2\alpha x}$$

and so when $H \neq 0$

$$\begin{aligned}
\frac{\partial^2\phi}{\partial x^2} &= 2e^{\phi}(H^2 - e^{-2\phi+2\alpha x}) \\
&= 2H^2e^{\phi} - 2e^{-\phi+2\alpha x} \\
&= 2|H|e^{\alpha x}(|H|e^{\phi-\alpha x} - \frac{1}{|H|}e^{-\phi+\alpha x}) \\
&= 2|H|e^{\alpha x}(e^{\phi-\alpha x+\log|H|} - e^{-\phi+\alpha x-\log|H|})
\end{aligned}$$

Letting $\eta(x) = \phi - \alpha x + \log|H|$ we find that $\eta(x)$ satisfies the differential equation

$$\eta''(x) = e^{\alpha x + \log|4H|} \sinh \eta(x) \tag{11 1}$$

and $\alpha = 0$ or 1 We note that this differential equation appears in Smyth [6] but with a minus sign Using (7 3) we have

$$\begin{aligned}
f_{xy} &= \alpha f_y + (\mathbf{E} + \alpha H f) * f_x \\
\langle f_{xy}, f_y \rangle_1 &= \alpha \langle f_y, f_y \rangle_1 + \langle (\mathbf{E} + \alpha H f) * f_x, f_y \rangle_1 \\
\frac{1}{2}(\langle f_y, f_y \rangle_1)_x &= \alpha e^{\phi} + \langle (\mathbf{E} + \alpha H f), f_x * f_y \rangle_1 \\
\frac{1}{2}(e^{\phi})_x &= \alpha e^{\phi} + e^{\phi} \langle (\mathbf{E} + \alpha H f), e^{-\phi} f_x * f_y \rangle_1 \\
\frac{1}{2} e^{\phi} \phi_x &= \alpha e^{\phi} + e^{\phi} \langle (\mathbf{E} + \alpha H f), \xi \rangle_1 \\
\phi_x &= 2\alpha + 2\langle (\mathbf{E} + \alpha H f), \xi \rangle_1
\end{aligned}$$

and hence

$$\eta'(x) = \alpha + 2\langle (\mathbf{E} + \alpha H f), \xi \rangle_1$$

Chapter 12

Completeness

We recall from chapter 6 that we may assume M is a strip on the complex plane given by $x_1 < x < x_2$ where $(x_1, x_2) = (-\infty, 1), (-\pi/2, \pi/2)$ or $(-\infty, \infty)$ and hence $\phi(x)$ must be defined on all of this interval For every y let $\gamma_y(t)$ $\mathbb{R} \to \mathbf{C}$ be given by $\gamma_y(t) = t + iy$ We are interested in the length of this curve, $L(\gamma_y|_{(x,x_0)})$, from any point $t = x$ to any other $t = x_0$ To this end we have

$$\gamma_y(t) = d\gamma_y(\frac{\partial}{\partial t}) = \left.\frac{\partial}{\partial x}\right|_{t+iy}$$

and hence the length is given by

$$L(\gamma_y|_{(x,x_0)}) = \int_x^{x_0} \sqrt{g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right)}dt = \int_x^{x_0} e^{\phi/2}dt$$

in order for the metric to be complete we should have that the length from any point in M to the boundary of M should be infinite and so $L(\gamma_y|_{(x_1,x)})$ and $L(\gamma_y|_{(x,x_2)})$ should be infinite for any finite $x \in (x_1, x_2)$

To get any further we first must analyze the differential equation (12 1) given below which we obtained in chapter 11

Problem

Classify the solutions to the differential equation (12 1)

$$\eta''(x) = e^{\alpha x + 2\beta} \sinh \eta(x) \tag{12 1}$$

where α equals 0 or 1 and $\beta \in \mathbb{R}$

In all that follows we shall assume $\eta(x)$ is a solution of (12 1)

Lemma 12.1

If $\eta(x)$ is not the trivial solution then it can have at most one critical point Moreover a critical point must either be a positive minimum or a negative maximum

Proof :

Consider the case when $x = c$ is a critical point of $\eta(x)$ i e $\eta'(c) = 0$ We must have $\eta(c)$ satisfying one of the following

(i) $\eta(c) = 0$,

$\eta(x) \equiv 0$ due to uniqueness of solutions

(ii.) $\eta(c) > 0$,

$\eta''(c) = e^{\alpha x + 2\beta} \sinh(\eta(c)) > 0$, and therefore $\eta(c)$ is a local minimum of $\eta(x)$ It follows η cannot have another critical point and consequently η has a global minimum at c

(iii.) $\eta(c) < 0$,

$\eta''(c) = e^{\alpha x + 2\beta} \sinh(\eta(c)) < 0$, and therefore $\eta(c)$ is a local maximum of η and as in (ii) must be a global maximum

Lemma 12.2

If x_0 is a point at which η is defined with $\eta(x_0) > 0$ and $\eta'(x_0) > 0$ then there exists a finite number $b > x_0$ such that $\eta(x) \to \infty$ as $x \to b$

Proof :

Since $\eta(x_0) > 0$ and $\eta'(x_0) > 0$ it follows from lemma 12 1 that $\eta(x) > 0$ for all $x > x_0$ at which $\eta(x)$ is defined

For every $x > x_0$ at which $\eta(x)$ is defined we shall examine $\eta''(x)$

$$\begin{aligned}\eta''(x) &= e^{\alpha x+2\beta}\sinh\eta(x)\\ &= e^{\alpha x+2\beta}(\eta(x)+\eta^3(x)/3!+\eta^5(x)/5!+\ \ )\\ &> e^{2\beta_1}\eta^3(x) \qquad\qquad (\text{as } \eta(x)>0)\end{aligned}$$

where $\beta_1 = \alpha x_0/2+\beta+\ln(6)/2$ Choose a positive real number r such that $0 < r < min\{\sqrt{\eta(x_0)}, \sqrt[4]{e^{-\beta_1}\sqrt{2}\eta'(x_0)}\}$ Then

$$0 < r^2 < \eta(x_0) \quad \text{and} \quad 0 < r^4(\frac{e^{\beta_1}}{\sqrt{2}}) < \eta'(x_0)$$

Now let $g(x)$ be the solution to the differential equation

$$g''(x) = e^{2\beta_1}g^3(x) \tag{12 2}$$

with initial conditions

$$g(x_0) = r^2 \quad \text{and} \quad g'(x_0) = \frac{r^4 e^{\beta_1}}{\sqrt{2}}$$

From Appendix F we can see that the solution is

$$g(x) = \frac{\sqrt{2}}{e^{\beta_1}(d-x)}, \qquad d = x_0 + \frac{\sqrt{2}}{e^{\beta_1}r^2}$$

We note that d is finite and that $g(x) > 0$ for all $x < d$ Moreover $g(x) \to \infty$ as $x \to d$ Comparing the two problems we see that

$$\eta''(x) > g''(x) \quad \eta'(x_0) > g'(x_0) \quad \eta(x_0) > g(x_0)$$

and consequently $\eta(x) > g(x)$ for all $x > x_0$ Since $g(x) \to \infty$ as $x \to d$ it is obvious that $\eta(x) \to \infty$ as $x \to b$ where $x_0 < b \leq d$ This concludes the proof of the lemma

From here on we consider the cases when $\alpha = 0$ and $\alpha = 1$ separately

Lemma 12.3 *($\alpha = 0$)*

Let $x_0 \in \mathbb{R}$ be a point at which η is defined with $\eta(x_0) > 0$ Then there exists unique real numbers $\gamma_l < 0 < \gamma_u$ such that

- *if $\eta'(x_0) > \gamma_u$*

  $\eta(x)$ is a strictly increasing function defined on a finite interval (a, b)

- *if $\eta'(x_0) = \gamma_u$*

  $\eta(x)$ is a strictly increasing function defined on the semi-infinite interval $(-\infty, b), b \in \mathbb{R}$ with $\eta(x) \to 0$ as $x \to -\infty$

- *if $\gamma_l < \eta'(x_0) < \gamma_u$*

  $\eta(x)$ has one critical point - a positive minimum and is defined on a finite interval (a, b)

- *if $\eta'(x_0) = \gamma_l$*

  $\eta(x)$ is a strictly decreasing function defined on a semi-finite interval $(a, \infty), a \in \mathbb{R}$ with $\eta(x) \to 0$ as $x \to \infty$

- *if $\eta'(x_0) < \gamma_l$*

  $\eta(x)$ is strictly decreasing defined on a finite interval (a, b)

Proof:

We first divide the solutions of (12 1) into three different categories

(a) $\eta(x)$ has critical points

(b) $\eta(x)$ has no critical points but there exists a point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$

(c) $\eta(x)$ has no critical points and there is no point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$

(a) By lemma 12 1 we know $\eta(x)$ has at most one critical point and it is either a positive minimum or a negative maximum Since $\eta(x_0) > 0$ the critial point must be a positive minimum, say it occurs at $x = x_1$ Let $h_1(x) = \eta(x + x_1)$ then

$$h_1''(x) = \eta''(x + x_1) = e^{2\beta} \sinh \eta(x + x_1) = e^{2\beta} \sinh h_1(x)$$

i e $h_1(x)$ satisfies the exact same differential equation with

$$h_1(0) = \eta(x_1) \quad \text{and} \quad h_1'(0) = \eta'(x_1) = 0$$

hence by replacing $\eta(x)$ with $\eta(x + x_1)$ we may assume that the minimum occurs at $x = 0$ For any $x > 0$ at which $\eta(x)$ is defined it is obvious that $\eta(x) > 0$ and $\eta'(x) > 0$ and hence by lemma 12 3 there exists some finite number b with $\eta(x) \to \infty$ as $x \to b$ Now let $h_2(x) = \eta(-x)$ then

$$h_2''(x) = \eta''(-x) = e^{2\beta} \sinh \eta(-x) = e^{2\beta} \sinh h_2(x)$$

and again $h_2(x)$ satisfies the exact same differential equation with

$$h_2(0) = \eta(0) \quad \text{and} \quad h_2'(0) = \eta'(0) = 0$$

since we are guaranteed uniqueness of solutions we know $h_2(x) \equiv \eta(x)$ Hence $\eta(x)$ is symmetric about the y-axis and hence $\eta(x) \to \infty$ as $x \to -b$ In the general case $\eta(x)$ is symmetric about the line $x = x_1$, its minimum, and has singularities at $x = x_1 \pm b$ $\triangle$

(b) $\eta(x)$ has no critical points but there exists a point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$ As in part (a) if we replace $\eta(x)$ with $\eta(x + x_1)$ we may assume that $x_1 = 0$ i e $\eta(0) = 0$ Since $\eta(x)$ has no critical points it must be either strictly increasing or strictly decreasing Replacing $\eta(x)$ with $\eta(-x)$ if necessary we may assume that the function is strictly increasing Hence, at any $x > 0$ where η is defined we must have that $\eta(x) > 0$ and also that $\eta'(x) > 0$ and by lemma 12 2 there must exist a finite number b such that $\eta(x) \to \infty$ as $x \to b$ Letting $h_4(x) = -\eta(-x)$ we see

$$h_4''(x) = -\eta''(-x) = -e^{2\beta}\sinh\eta(-x) = e^{2\beta}\sinh h_4(x)$$

with

$$h_4(0) = -\eta(0) = 0 \quad \text{and} \quad h_4'(0) = \eta'(0)$$

By uniqueness we must have $h_4(x) \equiv \eta(x)$ and hence we find that $\eta(x) \to -\infty$ as $x \to -b$ Hence, in the general case if η is a strictly increasing function then $\eta(x)$ is symmetric about x_1 with $\eta(x) \to \infty$ as $x \to x_1 + b$ and $\eta(x) \to -\infty$ as $x \to x_1 - b$ If η is a strictly decreasing function then $\eta(x)$ is symmetric about x_1 with $\eta(x) \to \infty$ as $x \to x_1 - b$ and $\eta(x) \to -\infty$ as $x \to x_1 + b$

$\triangle$

(c) $\eta(x)$ has no critical points and there is no point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$ Having no point x_1 at which $\eta(x_1) = 0$ means that the function is strictly positive or strictly negative and since $\eta(x_0) > 0$ we must have that η is strictly positive Having no critical points means $\eta(x)$ is either strictly increasing or strictly decreasing Let x_2 be a point at which η is defined Replacing $\eta(x)$ with $\eta(x + x_2)$ we may assmue that $x_2 = 0$ Then replacing $\eta(x)$ with $\eta(-x)$ if nessecary we may further assume that η is strictly decreasing Since η is both strictly positive and strictly decreasing it must exist over the entire interval $[0, \infty)$

If $\lim_{x\to\infty}\eta(x) = c \in \mathbf{R}/\{0\}$ then $\lim_{x\to\infty}\eta''(x) = \lim_{x\to\infty}\sinh(\eta(x)) = d \neq 0$ Therefore, $\eta'(x) \to \pm\infty$ as $x \to \infty$ and consequently $\eta(x) \to \pm\infty$ as $x \to \infty$ which is a contadiction Hence $\eta(x) \to 0$ or $\pm\infty$ as x tends to infinity As we may assume that η is both strictly positive and strictly decreasing we must have $\lim_{x\to\infty}\eta(x) = 0$

If we now let $u(x) = \eta(1/x)$ then,

$$
\begin{aligned}
u'(x) &= \eta'(\frac{1}{x})\,(-x^{-2}) \\
&= -\frac{1}{x^2}\,\eta'(\frac{1}{x}) \\
u''(x) &= 2x^{-3}\,\eta'(\frac{1}{x}) - \frac{1}{x^2}\,\eta''(\frac{1}{x})\,(-x^{-2}) \\
&= -\frac{2}{x}\,[-\frac{1}{x^2}\,\eta'(\frac{1}{x})] + \frac{1}{x^4}\,[\eta''(\frac{1}{x})] \\
&= -\frac{2}{x}\,u'(x) + \frac{1}{x^4}\,e^{2\beta}\sinh\eta(\frac{1}{x}) \\
&= -\frac{2}{x}\,u'(x) + \frac{1}{x^4}\,e^{2\beta}\sinh u(x)
\end{aligned}
$$

thus

$$
\begin{aligned}
u''(x) + \frac{2}{x}\,u'(x) &= \frac{1}{x^4}\,e^{2\beta}\sinh u(x) \\
x^2u''(x) + 2x\,u'(x) &= \frac{1}{x^2}\,e^{2\beta}\sinh u(x) \\
(x^2u'(x))' &= \frac{1}{x^2}\,e^{2\beta}\sinh u(x)
\end{aligned}
$$

$$
\begin{aligned}
2x^2u'(x)(x^2u'(x))' &= 2\,u'(x)\,e^{2\beta}\sinh u(x) \\
((x^2u'(x))^2)' &= (2\,e^{2\beta}\cosh u(x))' \\
(x^2u'(x))^2 &= 2\,e^{2\beta}\cosh u(x) + c_1, \quad c_1 \in \mathbf{R}
\end{aligned}
$$

Now $u(x) \to 0$ as $x \downarrow 0$ because $\eta(x) \to 0$ as $x \to \infty$ Furthermore $x^2u'(x) = -\eta'(1/x) \to 0$ as $x \downarrow 0$ because $\eta'(x) \to 0$ as $x \to \infty$ Hence $c_1 = -2e^{2\beta}$ and so we have

$$
\begin{aligned}
[x^2u'(x)]^2 &= e^{2\beta}(2\cosh(u(x)) - 2) \\
&= e^{2\beta}(e^{u(x)} + e^{-u(x)} - 2) \\
&= e^{2\beta}[e^{u(x)/2} - e^{-u(x)/2}]^2 \\
&= [2e^{\beta}\sinh(u(x)/2)]^2
\end{aligned}
$$

giving

$$x^2u'(x) = \pm\, 2\, e^{\beta}\ \sinh(u(x)/2)$$

Now $\eta(x)$ and $\eta'(x)$ have opposite signs as η is a positive function decreasing to 0 Thus $u(x)$ and $u'(x)$ have the same sign as $u(x) = \eta(1/x)$ and $u'(x) = -1/x^{-2}\eta'(1/x)$ Thus

$$x^2u'(x) = 2\, e^{\beta}\ \sinh(u(x)/2)$$

and

$$\frac{1}{2}\int \frac{1}{\sinh(u/2)}du = e^{\beta}\int \frac{1}{x^2}dx$$

giving

$$\ln\left|\frac{\sinh(u(x)/4)}{\cosh(u(x)/4)}\right| = e^{\beta}(-\frac{1}{x} + c_2)$$

As η is positive

$$\tanh(u(x)/4) = e^{e^{\beta}(-1/x+c_2)}$$

thus

$$u(x) = 4\tanh^{-1}(e^{e^{\beta}(c_2-1/x)})$$

and

$$\eta(x) = 4\tanh^{-1}(e^{e^{\beta}(c_2-x)})$$

which has a singularity at $x = c_2 < 0$ Replacing $\eta(x)$ with $\eta(x - x_2)$ we arrive back at the general case when η is strictly decreasing and so

$$\eta(x) = 4\tanh^{-1}(e^{e^{\beta}(c-x)}) \tag{12 3}$$

for some $c \in \mathbb{R}$ We note that η is defined only on the interval (c, ∞) Also replacing $\eta(x)$ with $\eta(-x)$ we have

$$\eta(x) = 4\tanh^{-1}(e^{e^{\beta}(x-c)}) \tag{12 4}$$

the general case for when η is strictly increasing We note that here η is defined only on the interval $(-\infty, c)$

$\triangle$

To summarise, for any initial condition $\eta(x_0) = \nu > 0$ there is exactly one solution to (12 1) with $\alpha = 0$ of the form (12 4) call this $\eta_1(x)$ and one solution to (12 1) with $\alpha = 0$ of the form (12 3) call this $\eta_2(x)$ We note that these are the only solutions to (12 1) with $\alpha = 0$ of type (c) mentioned above Let $\gamma_u = \eta_1'(x_2)$ and $\gamma_l = \eta_2'(x_2)$ Obviously $\gamma_l < 0 < \gamma_u$

If $\eta(x)$ is another solution to (12 1) with $\alpha = 0$ and $\eta(x_0) = \nu$ then the solution must be of the form (a) or (b) defined above and hence one of the following must hold

1 $\eta'(x_0) > \gamma_u$ in which case, by lemma 12 4 given below, $\eta(x) > \eta_1(x)$ for all $x > x_0$ and $\eta(x) < \eta_1(x)$ for all $x < x_0$ Therefore $\eta(x)$ is of the form b defined above

2 $\eta'(x_0) < \gamma_l$ in which case, by lemma 12 4 given below, $\eta(x) < \eta_2(x)$ for all $x > x_0$ and $\eta(x) > \eta_2(x)$ for all $x < x_0$ Therefore $\eta(x)$ is also of the form b above

3 $\gamma_u > \eta'(x_0) > \gamma_l$ in which case $\eta(x) > \eta_2(x)$ for all $x > x_0$ and $\eta(x) > \eta_1(x)$ for all $x < x_0$ by lemma 12 4 and hence is of the form (a) above i e it has a positive minimum

This concludes the proof of the lemma

Lemma 12.4

Let $x_0, a, b, c \in \mathbb{R}$ with $b > c$ and let $\eta_1(x)$ and $\eta_2(x)$ be solutions to the differential equation

$$\eta'' = e^{2\beta} \sinh \eta$$

with

$$\eta_1(x_0) = a \qquad \eta_1'(x_0) = b$$

and

$$\eta_2(x_0) = a \qquad \eta_2'(x_0) = c$$

then $\eta_1(x) > \eta_2(x)$ for every $x > x_0$ at which both η_1 and η_2 are defined and $\eta_1(x) < \eta_2(x)$ for every $x < x_0$ at which both η_1 and η_2 are defined

Proof:

Since $\eta_1(x_0) = \eta_2(x_0)$ and $\eta_1'(x_0) > \eta_2'(x_0)$ there must exist some interval (x_0, x_1) in which in which both $\eta_1(x) > \eta_2(x)$ and $\eta_1'(x) > \eta_2'(x)$ Hence $\eta_1''(x) = e^{2\beta} \sinh \eta_1(x)$ $> e^{2\beta} \sinh \eta_2(x) = \eta_2''(x)$ for every $x \in (x_0, x_1)$ and so $\eta_1(x_1) > \eta_2(x_1)$, $\eta_1'(x_1) > \eta_2'(x_1)$ and $\eta_1''(x_1) > \eta_2''(x_1)$ (assuming $\eta_2(x)$ exists at x_1, if not then we have already completed the proof) We then may extend beyond x_1 to another interval (x_1, x_2) in which $\eta_1(x) > \eta_2(x)$, $\eta_1'(x) > \eta_2'(x)$ and $\eta_1''(x) > \eta_2''(x)$ and the process can continue so long as $\eta_1(x)$ and $\eta_2(x)$ are defined We use the same approach to prove $\eta_1(x) < \eta_2(x)$ for all x less than x_0 in which $\eta_1(x)$ and $\eta_2(x)$ are defined

Lemma 12.5 *($\alpha = 0$)*

Let $x_0 \in \mathbb{R}$ be a point at which η is defined with $\eta(x_0) < 0$ Then there exists unique real numbers $\gamma_l < 0 < \gamma_u$ such that

- *if $\eta'(x_0) > \gamma_u$*

  $\eta(x)$ is a strictly increasing function defined on a finite interval (a, b)

- *if* $\eta'(x_0) = \gamma_u$

  $\eta(x)$ *is strictly increasing function defined on the semi-infinite interval* $(-\infty, b), b \in \mathbb{R}$ *with* $\eta(x) \to 0$ *as* $x \to -\infty$

- *if* $\gamma_l < \eta'(x_0) < \gamma_u$

  $\eta(x)$ *has only one critical point - a negative maximum and is defined on the finite interval* (a, b)

- *if* $\eta'(x_0) = \gamma_l$

  $\eta(x)$ *is a strictly decreasing function defined on a semi-infinte strip* $(a, \infty), a \in \mathbb{R}$ *with* $\eta(x) \to 0$ *as* $x \to \infty$

- *if* $\eta'(x_0) < \gamma_l$

  $\eta(x)$ *is strictly decreasing and is defined on the finite strip* (a, b)

Proof:

The proof follows by replacing $y(x)$ with $-y(x)$ and then using lemma 12 3

We now examine the completeness of the metric when $\alpha = 0$ Here $\phi(x) = \eta(x) - \ln|H|$ and η satisfies the differential equation

$$\eta''(x) = 4|H| \sinh \eta(x)$$

Using lemmas 12 3 and 12 5 we can see that other than the solution $\phi(x) = -\ln|H|$, ϕ never exists over the entire real line Again using lemmas 12 3 and 12 5 we see there are exactly two solutions which exist over the interval $(-\infty, 1)$ which are given by

$$\phi(x) = \pm 4 \tanh^{-1}(e^{2\sqrt{|H|}(x-1)}) - \ln|H|$$

When

$$\phi(x) = -4 \tanh^{-1}(e^{2\sqrt{|H|}(x-1)}) - \ln|H|$$

we see $\phi \to -\infty$ as $x \uparrow 1$ and so $e^{\phi/2} \to 0$ as $x \uparrow 1$ hence the metric is not complete On the other hand when

$$\phi(x) = 4\tanh^{-1}(e^{2\sqrt{|H|}(x-1)}) - \ln|H|$$

$\phi(x) \to \infty$ as $x \uparrow 1$ and furthermore

$$e^{\phi(x)/2} = \frac{1}{\sqrt{|H|}} e^{2\tanh^{-1} e^{2\sqrt{|H|}(x-1)}} > \frac{1}{\sqrt{|H|}} e^{2\tanh^{-1} e^{-2\sqrt{|H|}}} \quad \frac{1}{1-x}$$

for all $0 < x < 1$ so that $\int e^{\phi/2}dx \to \infty$ as $x \uparrow 1$ As $x \to -\infty$ we have that $\eta(x) \to 0$ and hence $e^{\phi/2} \to \frac{1}{\sqrt{|H|}}$ It follows that $\int e^{\phi/2}dx \to \infty$ as $x \to -\infty$ and hence the metric is complete

This leaves us with the case when ϕ is defined on the interval (a, b) As we are looking for the metric to be complete we want $\phi \to \infty$ as $x \to a$ and $\phi \to \infty$ as $x \to b$ From lemmas 12 3 and 12 5 we see that there is only one possibility that η has a positive minimum From the proof of lemma 12 3 we recall that if η has a positive minimum then η is symmetric about its minimum Hence in order that $\eta(x) \to \infty$ as $x \to a$ and b we must have $\eta'(\frac{a+b}{2}) = 0$ The question now arises as to whether these solutions give rise to ϕ being complete Hence let η_u be a solution to the differential equation

$$\eta''(x) = 4|H| \sinh \eta(x)$$

which has a positive minimum and singularities at some values $x = a$ and $x = b$ Also let $\eta_l(x)$, be the solution to the differential equation

$$\eta''(x) = 4|H| \sinh \eta$$

with

$$\eta(x) \to 0 \ \text{ as } \ x \to -\infty \ \text{ and } \ \eta(x) \to \infty \ \text{ as } \ x \to b$$

i e let

$$\eta_l(x) = 4\tanh^{-1} e^{2\sqrt{|H|}(x-b)}$$

Now $\eta_u(x)$ and $\eta_l(x)$ both tend to infinity as x tends to b and $\eta_u'(\frac{a+b}{2}) = 0 < \eta_l'(\frac{a+b}{2})$ so that $\eta_u(\frac{a+b}{2}) > \eta_l(\frac{a+b}{2})$ It follows that $\eta_u(x) > \eta_l(x)$ for all $x > \frac{a+b}{2}$ and hence $\int e^{\phi/2}dx \to \infty$ as $x \to b$ Now ϕ is symmetric about $\frac{a+b}{2}$ since $\eta_u(x)$ is and so $\int e^{\phi/2}dx \to \infty$ as $x \to a$ Hence, the metric is complete

We now study the differential equation (12 1) with $\alpha = 1$ Letting $g(x) = \eta(x-2c)$ we observe that $g'(x) = \eta'(x-2c)$ and

$$\begin{aligned} g''(x) &= \eta''(x-2c) = e^{x-2c+2\beta}\sinh\eta(x-2c) \\ &= e^{x+2(\beta-c)}\sinh g(x) \end{aligned}$$

So that $g(x)$ satisfies the same differential equation as η with only the value of β changing Hence from here on we shall assume that η satisfies the differential equation

$$\eta''(x) = 4e^x \sinh\eta$$

Lemma 12.6 *($\alpha = 1$)*

Let $x_0, a, b, c \in \mathbb{R}$ with $a > c$ and let $\eta_1(x)$ and $\eta_2(x)$ be solutions to the differential equation

$$\eta'' = 4e^x \sinh\eta$$

with

$$\eta_1(x_0) = a \qquad \eta_1'(x_0) = b$$

and

$$\eta_2(x_0) = c \qquad \eta_2'(x_0) = b$$

then $\eta_1(x) > \eta_2(x)$ for every x at which both η_1 and η_2 are defined

Proof.

Since $\eta_1''(x_0) = 4e^{x_0}\sinh\eta_1(x_0) > 4e^{x_0}\sinh\eta_2(x_0) = \eta_2''(x_0)$ and $\eta_1(x_0) > \eta_2(x_0)$ there must exist some interval (x_0, x_1) in which both $\eta_1(x) > \eta_2(x)$ and $\eta_1''(x) > \eta_2''(x)$ and since $\eta_1'(x_0) = \eta_2'(x_0)$ we must have that $\eta_1'(x) > \eta_2'(x)$ in this interval Hence $\eta_1(x_1) > \eta_2(x_1)$, $\eta_1'(x_1) > \eta_2'(x_1)$ and $\eta_1''(x_1) > \eta_2''(x_1)$ (assuming both $\eta_1(x)$ and $\eta_2(x)$ exist at x_1, if not then we have already completed the proof) We then may extend beyond x_1 to another interval (x_1, x_2) in which $\eta_1(x) > \eta_2(x)$, $\eta_1'(x) > \eta_2'(x)$ and $\eta_1''(x) > \eta_2''(x)$ and the process can continue so long as both $\eta_1(x)$ and $\eta_2(x)$ are defined We use the same approach to prove $\eta_1(x) > \eta_2(x)$ for all x less than x_0 in which both $\eta_1(x)$ and $\eta_2(x)$ are defined

Lemma 12 7

Let $x_0, a, b, c \in \mathbb{R}$ with $b > c$ and let $\eta_1(x)$ and $\eta_2(x)$ be solutions to the differential equation

$$\eta'' = 4e^x \sinh\eta$$

with

$$\eta_1(x_0) = a \qquad \eta_1'(x_0) = b$$

and

$$\eta_2(x_0) = a \qquad \eta_2'(x_0) = c$$

then $\eta_1(x) > \eta_2(x)$ for every $x > x_0$ at which both η_1 and η_2 are defined and $\eta_1(x) < \eta_2(x)$ for every $x < x_0$ at which both η_1 and η_2 are defined

Proof.

Since $\eta_1(x_0) = \eta_2(x_0)$ and $\eta_1'(x_0) > \eta_2'(x_0)$ there must exist some interval (x_0, x_1) in which in which both $\eta_1(x) > \eta_2(x)$ and $\eta_1'(x) > \eta_2'(x)$ Hence $\eta_1''(x) = 4e^x \sinh\eta_1(x) > 4e^x \sinh\eta_2(x) = \eta_2''(x)$ and so $\eta_1(x_1) > \eta_2(x_1)$, $\eta_1'(x_1) > \eta_2'(x_1)$ and $\eta_1''(x_1) > \eta_2''(x_1)$ (assuming $\eta_2(x)$ exists at x_1, if not then we have already completed the proof) We then may extend beyond x_1 to another interval (x_1, x_2) in which $\eta_1(x) > \eta_2(x)$, $\eta_1'(x) > \eta_2'(x)$ and $\eta_1''(x) > \eta_2''(x)$ and the process can continue so long as $\eta_1(x)$ and $\eta_2(x)$ are defined We use the same approach to prove $\eta_1(x) < \eta_2(x)$ for all x less than x_0 in which $\eta_1(x)$ and $\eta_2(x)$ are defined

Lemma 12 8

If $\eta_1(x)$ is a solution of

$$\eta'' = 4e^x \sinh \eta$$

with

$$\eta(x_0) = a \qquad \eta'(x_0) = b$$

and $\eta_1(x)$ satisfies all the following properties

- *has one critical point - a positive minimum*
- *has a singularity at some finite value $x_1 < 0$*

then there is no solution to the differential equation with $\eta(x_0) \geq a$ which exists over the whole real line

Note any solution to the differential equation which has the same properties as η_1 shall be called a solution of type U Also by lemma 12 2 every solution of type U has a singularity at some point $x_2 > 0$

Proof:

We first show that there is no solution, $\eta_2(x)$ to the differential equation (12 1) with $\eta_2(x_0) = a$ which exists over the whole real line

If $\eta_2'(x_0) = b$ then $\eta_2(x) \equiv \eta_1(x)$ by uniqueness and hence does not exist over the whole real line If $\eta_2'(x_0) > b$ then by lemma 12 7 $\eta_2(x) > \eta_1(x)$ for all x greater than x_0 and since $\eta_1(x)$ has a singularity at some x greater than zero so must $\eta_2(x)$ Similarly when $\eta_2'(x_0) < b$ we have that $\eta_2(x)$ has a singularily at some $x < 0$

To prove the statement when $\eta_2(x_0) > a$ First let $\eta_3(x)$ be another solution to the differential equation with $\eta_3(x_0) = \eta_2(x_0) > a$ and $\eta_3'(x_0) = b$ then by lemma 12 7 $\eta_3(x)$ satisifies the same three conditions that $\eta_1(x)$ satisfies Then using the approach used in the previous paragraph we can show that $\eta_2(x)$ cannot exist over the entire real line This completes the proof

We now study more closely the properties of the solutions to the differential equation

We now note that $\eta(x) \equiv 0$ is a solution of the differential equation and from here on we shall assume η is non-trivial Letting $\tilde{\eta} = -\eta(x)$ we note that $\tilde{\eta}'(x) = -\eta'(x)$ and $\tilde{\eta}'' = -\eta'' = -4e^x \sinh \eta = 4e^x \sinh -\eta = 4e^x \sinh \tilde{\eta}$ i e $\tilde{\eta}$ is also a solution of the differential equation Thus by replacing $\eta(x)$ with $-\eta(x)$ if necessary we may assume in all that follows that $\eta(x_0) \geq 0$ As we are dealing with non-trivial solutions we may further assume $\eta(x_0) > 0$

In view of lemma 12 1 we remark that each solution $\eta(x)$ must satisfy one of the following conditions

a $\eta(x)$ has a positive minimum

b $\eta(x)$ has no critical points but there exists a point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$

c $\eta(x)$ has no critical points and there is no point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$

Solutions of type a and b exist for every initial condition $\eta(x_0) > 0$ and by lemma 12 2 these solutions will tend to ∞ as x tends to some finite number $b > x_0$ Hence if a solution exists on an interval $[x_0, \infty)$ then it must be of type c above i e it must be either strictly increasing or strictly decreasing and given $\eta(x_0) > 0$ it must also be strictly positive Again by lemma 12 2 a strictly increasing solution would tend to infinity as x tended to some finite $b > x_0$ hence we have that η is stricly decreasing If $\lim_{n\to\infty} \eta(x) = d > 0$ then $\eta''(x) = 4e^x \sinh \eta(x)$ tends to infinty as x tends to infinity which in turn implys $\eta'(x)$ and $\eta(x)$ would tend to infinity as

x tends to infinity - a contradiction Hence $\lim_{n\to\infty} \eta(x) = 0$ In summary, every solution of type c which exists over the interval $[x_0, \infty)$ must be a strictly positive and strictly decreasing and tend to zero as x tends to infinity

Lemma 12 9

If x_1 is a point at which η is defined with $\eta(x_1) > 0$ and $\eta'(x_1) < 0$ then $\eta(x) \to \infty$ as $x \to b$ where $b = -\infty$ or is some finite number less than x_1

Proof

Let η be defined on the region (b, x_1) where $b = -\infty$ or is just some finite number less than x_1 As η is of type a b or c above - it is obvious that $\eta(x) > 0$ for all $x \in (b, x_1)$ Now let $g(x)$ be the solution to the differential equation

$$g''(x) = 0 \quad g'(x_1) = \eta'(x_1) \quad g(x_1) = \eta(x_1)$$

so that $g(x) = \eta'(x_1)x + \eta(x_1)$ Comparing $\eta(x)$ to $g(x)$ we see $\eta''(x) > g''(x)$ for all $x \in (b, x_1)$, $\eta'(x) = g'(x_1)$ and $\eta(x_1) = g(x_1)$ so we must have $\eta(x) > g(x)$ for all $x \in (b, x_1)$ Since $g(x)$ tends to ∞ as x tends to ∞ we must have $\eta(x)$ tending to ∞ as x tends to b

We now examine the completeness of these solutions

a $\eta(x)$ has a positive minimum Thus η is defined on the region (a, b) where b is some finite number and $a \in \mathbb{R}$ or $a = -\infty$ Also $\eta(x)$ tends to infinity as x tends to both a and b Hence $e^{\eta(x)+x+c}$ tends to infinity as x tends to both a and b Thus depending on the initial conditions - the surfaces could be complete

b $\eta(x)$ has no critical points but there exists a point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$ Let η be defined on the region (a, b) where $a \in \mathbb{R}$ or $a = -\infty$ and $b \in \mathbb{R}$ or $b = \infty$ If we assume η is strictly decreasing then by lemma 12 2 b would be finite and

$\eta(x) \to -\infty$ as $x \to b$ Hence $e^{\eta+x+c} \to 0$ as $x \to b$ and the solution would not be complete If we assume that $\eta(x)$ is strictly increasing then $\eta(x)$ is negative for all $x \in (a, x_1)$ and hence $e^{\eta+x+c}$ tends to zero as x tends to a Hence if the solution to the differential equation is of type b then the surface will not be complete

c $\eta(x)$ has no critical points and there is no point $x_1 \in \mathbb{R}$ such that $\eta(x_1) = 0$ Let η be defined on the region (a, b) where $a \in \mathbb{R}$ or $a = -\infty$ and $b \in \mathbb{R}$ or $b = \infty$ Since we may assume $\eta(x_0) > 0$ we have that η is stricly positive If it is strictly increasing then $\lim_{x \to a} \eta(x) = d \in \mathbb{R}$ and hence $e^{\eta+x+c} \to 0$ as $x \to a$ i e the surface is not complete On the other hand if η is strictly decreasing we know it must exist on the interval $[x_0, \infty)$ and $\eta(x) \to 0$ as $x \to \infty$ Also by lemma 12 9 we have that $\eta(x) \to \infty$ as $x \to a$ and a may be finite or equal $-\infty$ We recall from chapter 6 that we are only interested in solutions which exist over the intervals $(-\infty, \infty), (-\infty, 1)$ or (c, d) where c and d are finite So if a solution exists on the interval $[x_0, \infty)$ it is clear that the it would be finite at $x = 1$ or $x = d$ (assuming it exists there) and hence would lead to the surface not being complete So we are only interested in solutions with $a = -\infty$ Hence if a solution of type c gives rise to a complete surface we must have that the solution exists over the entire real line with $\eta(x) \to 0$ as $x \to \infty$ and $\eta(x) \to \infty$ as $x \to -\infty$ If such a solution exists it is clear that $e^{\eta+x+c} \to \infty$ as $x \to \pm\infty$

In summary, any solution to the differential equation (12 1) which gives rise to a complete metric must either

i have a positive minimum and is defined on a finite interval (a, b)

ii have a positive minimum and is defined on a semi-infinite interval $(-\infty, b)$

iii be a strictly positive, strictly decreasing function which tends to 0 as x tends to ∞ and to ∞ as x tends to $-\infty$

We note that the existence of solutions of type ii and iii have not been proved

We do however recall lemma 12 8 which states that if a solution of type i occurs with initial condition $\eta(x_0) = \gamma > 0$ then a solution of type iii cannot occur with initial condtion $\eta(x_0) = \gamma$, the converse of this is also true I would conjecture that no solutions of type iii occur Letting $g(x) = \eta(1/x)$ this conjecture is equivalent to stating that the following differential equation

$$(x^2 g'(x))^2 = \frac{4}{x^2} e^{1/x} \sinh g(x)$$

with the mixed boundary conditons

$$g(1) > 0 \quad \lim_{x \to 0} g(x) = 0$$

has no solutions

Chapter 13

Minimal Surfaces

From here on we shall drop the subscript 1 from the inner product symbol i e

$$\langle x, y\rangle = \langle x, y\rangle_1$$

and subscript 1 will imply partial differentiation with respect to x, similarly subscript 2 will imply differentiation with respect to y

We first study the minimal surfaces when $\alpha \neq 0$ Replacing α with $-\alpha$ we have from equations (7 3) (7 4) and (7 5) that f satisfies the following

$$f_{12} = -\alpha f_2 + \mathbf{E} * f_1 \qquad (13\ 1)$$

$$f_{22} = \alpha f_1 + \mathbf{E} * f_2 \qquad (13\ 2)$$

$$f_{11} = -f_{22} \qquad (13\ 3)$$

If we assume that $\mathbf{E}$ is the zero vector then these equations reduce to

$$f_{12} = -\alpha f_2 \qquad (13\ 4)$$

$$f_{22} = \alpha f_1 \qquad (13\ 5)$$

$$f_{11} = -\alpha f_1 \qquad (13\ 6)$$

(13 4) and (13 6) imply

$$f_1 = -\alpha f + v_1$$

for some $v_1 \in \mathbb{R}^3$ This has solution

$$f(x, y) = e^{-\alpha x} v_2(y) + \frac{v_1}{\alpha}$$

(13 5) then implies

$$v_2(y) = v_3\cos(\alpha y) + v_4\sin(\alpha y)$$

for some $v_3, v_4 \in \mathbb{R}^3$, and so f is planar Hence from here on we shall assume $\mathbf{E}$ is not the zero vector We let $\epsilon_1 = \langle \mathbf{E}, \mathbf{E} \rangle$ We recall from lemma 9 2 that

$$\langle \mathbf{E}, f \rangle = a\cos(\alpha y + t)e^{-\alpha x}$$

By switching to an associate if necessary we may assume $t = 0$ and hence that

$$\langle \mathbf{E}, f_y(0,0) \rangle = 0 \tag{13 7}$$

and

$$\langle \mathbf{E}, f_x(0,0) \rangle = -\alpha a \tag{13 8}$$

Lemma 13.1

After a translation orthogonal to $\mathbf{E}$ *we may assume without loss of generality that*

$$\mathbf{E} * f_2 = \alpha f_1 + (\alpha^2 + \epsilon_1)f - \langle \mathbf{E}, f \rangle \mathbf{E} + c$$

where $c \in \mathbb{R}^3$ *and* $\langle \mathbf{E}, c \rangle = 0$ *Futhermore we may assume* $c = (0,0,0)^T$ *if* $\epsilon_1 \neq -\alpha^2$

Proof .

$$(13\ 1) \Rightarrow \quad f_{12} = -\alpha f_2 + \mathbf{E} * f_1$$

$$\begin{aligned}
\mathbf{E} * f_{12} &= -\alpha \mathbf{E} * f_2 + \mathbf{E} * (\mathbf{E} * f_1) \\
&= -\alpha(f_{22} - \alpha f_1) + \mathbf{E} * (\mathbf{E} * f_1) \\
&= \alpha f_{11} + \alpha^2 f_1 + \mathbf{E} * (\mathbf{E} * f)_1
\end{aligned}$$

$$\mathbf{E} * f_2 = \alpha f_1 + \alpha^2 f + \mathbf{E} * (\mathbf{E} * f) + \psi(y)$$

Also

$$(13\ 2) \Rightarrow \quad f_{22} = \alpha f_1 + \mathbf{E} * f_2$$

$$\begin{aligned}
\mathbf{E} * f_{22} &= \alpha \mathbf{E} * f_1 + \mathbf{E} * (\mathbf{E} * f_2) \\
&= \alpha(f_{12} + \alpha f_2) + \mathbf{E} * (\mathbf{E} * f_2) \\
&= \alpha f_{12} + \alpha^2 f_2 + \mathbf{E} * (\mathbf{E} * f)_2
\end{aligned}$$

$$\mathbf{E} * f_2 = \alpha f_1 + \alpha^2 f + \mathbf{E} * (\mathbf{E} * f) + \tilde{\psi}(x)$$

Thus

$$\psi(y) = \tilde{\psi}(x) = c \in \mathbb{R}^3$$

and

$$\mathbf{E} * f_2 = \alpha f_1 + \alpha^2 f + \mathbf{E} * (\mathbf{E} * f) + c \tag{13 9}$$

As $\langle \mathbf{E}, f \rangle = a \cos(\alpha y + t) e^{-\alpha x}$ we have

$$\langle \mathbf{E}, f_1 \rangle = \langle \mathbf{E}, f \rangle_1 = -\alpha \langle \mathbf{E}, f \rangle \tag{13 10}$$

and we have

$$\begin{aligned} 0 &= \langle \mathbf{E}, \mathbf{E} * f_2 \rangle \\ &= \alpha\langle \mathbf{E}, f_1\rangle + \alpha^2\langle \mathbf{E}, f\rangle + \langle \mathbf{E}, \mathbf{E} * (\mathbf{E} * f)\rangle + \langle \mathbf{E}, c\rangle \quad \text{by (13 9)} \\ &= \langle \mathbf{E}, c\rangle \quad \text{by (13 10)} \end{aligned}$$

that is, c is orthogonal to $\mathbf{E}$ and

$$\mathbf{E} * (\mathbf{E} * c) = \langle \mathbf{E}, \mathbf{E}\rangle c - \langle \mathbf{E}, c\rangle \mathbf{E} = \epsilon_1 c$$

Now if $\epsilon_1 \neq -\alpha^2$ replace f by $\tilde{f} = f + \frac{c}{\alpha^2+\epsilon_1}$ then

$$\begin{aligned} (13\ 9) \Rightarrow \mathbf{E} * \tilde{f}_2 &= \mathbf{E} * f_2 \\ &= \alpha f_1 + \alpha^2 f + \mathbf{E} * (\mathbf{E} * f) + c \\ &= \alpha\tilde{f}_1 + \alpha^2\tilde{f} - \frac{\alpha^2 c}{\alpha^2+\epsilon_1} + \mathbf{E} * (\mathbf{E} * \tilde{f}) - \mathbf{E} * (\mathbf{E} * \frac{c}{\alpha^2+\epsilon_1}) + c \\ &= \alpha\tilde{f}_1 + \alpha^2\tilde{f} + \mathbf{E} * (\mathbf{E} * \tilde{f}) - \frac{\alpha^2 c}{\alpha^2+\epsilon_1} - \frac{1}{\alpha^2+\epsilon_1}\epsilon_1 c + c \\ &= \alpha\tilde{f}_1 + \alpha^2\tilde{f} + \mathbf{E} * (\mathbf{E} * \tilde{f}) \end{aligned}$$

Thus we have now shown that if $\epsilon_1 \neq -\alpha^2$ then we may assume $c = (0,0,0)^T$ Hence after a translation orthogonal to $\mathbf{E}$ we may assume

$$\begin{aligned} f &= \alpha f_1 + \alpha^2 f + \mathbf{E} * (\mathbf{E} * f) + c \\ &= \alpha f_1 + \alpha^2 f + \langle \mathbf{E}, \mathbf{E}\rangle f - \langle \mathbf{E}, f\rangle \mathbf{E} + c \end{aligned}$$

proving the lemma

We shall now divide our analysis into four cases

1. $||\mathbf{E}||^2 = \epsilon^2$
2. $||\mathbf{E}||^2 = -\epsilon^2 \neq -\alpha^2$
3. $||\mathbf{E}||^2 = -\alpha^2$
4. $||\mathbf{E}||^2 = 0$

for some $\epsilon > 0$

Lemma 13 2

If $||\mathbf{E}||^2 = \epsilon^2 > 0$ *then we may assume*

$$f(x,y) = e^{-\alpha x}(\mathbf{U}(y)\cos(\epsilon x) + \mathbf{V}(y)\sin(\epsilon x) + \frac{a}{\epsilon^2}\cos(\alpha y + t)\mathbf{E})$$

where $\langle \mathbf{U}(y), \mathbf{E}\rangle = \langle \mathbf{V}(y), \mathbf{E}\rangle = 0$

and if $||\mathbf{E}||^2 = -\epsilon^2 < 0$ *where* $\epsilon^2 \neq \alpha^2$ *then after a translation in* $\mathbb{R}^3$ *we may assume*

$$f(x,y) = \mathbf{U}(y)e^{(-\alpha+\epsilon)x} + \mathbf{V}(y)e^{(-\alpha-\epsilon)x} - \frac{a}{\epsilon^2}e^{-\alpha x}\cos(\alpha y + t)\mathbf{E}$$

where $\langle \mathbf{U}(y), \mathbf{E}\rangle = \langle \mathbf{V}(y), \mathbf{E}\rangle = 0$

Proof :

In both cases we have using lemma 13 1 that

$$\begin{aligned}(13\ 3) \Rightarrow f_{11} &= -f_{22} \\ &= -(\alpha f_1 + \mathbf{E} * f_2) \\ &= -(2\alpha f_1 + (\alpha^2 + \epsilon_1)f - \langle \mathbf{E}, f\rangle \mathbf{E})\end{aligned}$$

therefore

$$f_{11} + 2\alpha f_1 + (\alpha^2 + \epsilon_1)f = a\cos(\alpha y + t)e^{-\alpha x}\mathbf{E} \tag{13 11}$$

for a particular solution of (13 11) we try

$$P(x,y) = \mathbf{W}(y)e^{-\alpha x}$$

and substituting this in, we get

$$\alpha^2\mathbf{W}(y)e^{-\alpha x} - 2\alpha^2\mathbf{W}(y)e^{-\alpha x} + (\alpha^2+\epsilon_1)\mathbf{W}(y)e^{-\alpha x} = ae^{-\alpha x}\cos(\alpha y+t)\mathbf{E}$$

$$\Rightarrow \mathbf{W}(y) = \frac{a}{\epsilon_1}\cos(\alpha y+t)\mathbf{E}$$

The homogenous equation

$$f_{11} + 2\alpha f_1 + (\alpha^2+\epsilon_1)f = 0$$

has characteristic equation

$$\lambda^2 + 2\alpha\lambda + (\alpha^2+\epsilon_1) = 0$$

which has roots

$$\begin{aligned}\lambda &= \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4(1)(\alpha^2+\epsilon_1)}}{2} \\ &= \frac{-2\alpha \pm \sqrt{-4\epsilon_1}}{2} \\ &= -\alpha \pm \sqrt{-\epsilon_1}\end{aligned}$$

So in the case when $\langle E,E\rangle = \epsilon^2 > 0$ we have

$$f(x,y) = \mathbf{U}(y)e^{-\alpha x}\cos(\epsilon x) + \mathbf{V}(y)e^{-\alpha x}\sin(\epsilon x) + (\text{a particular solution })$$

and so

$$f(x,y) = (\mathbf{U}(y)\cos\epsilon x + \mathbf{V}(y)\sin\epsilon x + \frac{a}{\epsilon^2}\cos(\alpha y+t)\mathbf{E})e^{-\alpha x}$$

Now take the innerproduct with $\mathbf{E}$ across this equation to give

$$\langle \mathbf{E}, f\rangle = (\langle \mathbf{U}(y), \mathbf{E}\rangle \cos(\epsilon x) + \langle \mathbf{V}(y), \mathbf{E}\rangle \sin(\epsilon x) + a\cos(\alpha y + t))e^{-\alpha x}$$

Note from lemma 1 $\langle \mathbf{E}, f\rangle = ae^{-\alpha x}\cos(\alpha y + t)$

$$0 = \langle \mathbf{U}(y), \mathbf{E}\rangle \cos(\epsilon x) + \langle \mathbf{V}(y), \mathbf{E}\rangle \sin(\epsilon x)$$

$$\Rightarrow \langle \mathbf{U}(y), \mathbf{E}\rangle \equiv \langle \mathbf{V}(y), \mathbf{E}\rangle \equiv 0$$

proving the lemma when $\langle \mathbf{E}, \mathbf{E}\rangle > 0$ When $\langle E, E\rangle = -\epsilon^2 < 0, \epsilon^2 \neq \alpha^2$ we have

$$f(x,y) = \mathbf{U}(y)e^{(-\alpha+\epsilon)x} + \mathbf{V}(y)e^{(-\alpha-\epsilon)x} + (\text{a particular solution })$$

and so

$$f(x,y) = \mathbf{U}(y)e^{(-\alpha+\epsilon)x} + \mathbf{V}(y)e^{(-\alpha-\epsilon)x} - \frac{a}{\epsilon^2}e^{-\alpha x}\cos(\alpha y + t)\mathbf{E}$$

again taking the innerproduct with $\mathbf{E}$ across this equation to give

$$\langle \mathbf{E}, f\rangle = \langle \mathbf{U}(y), \mathbf{E}\rangle e^{(-\alpha+\epsilon)x} + \langle \mathbf{V}(y), \mathbf{E}\rangle e^{(-\alpha-\epsilon)x} + ae^{-\alpha x}\cos(\alpha y + t)$$

again from lemma 1 $\langle \mathbf{E}, f\rangle = ae^{-\alpha x}\cos(\alpha y + t)$

$$0 = \langle \mathbf{U}(y), \mathbf{E}\rangle e^{(-\alpha+\epsilon)x} + \langle \mathbf{V}(y), \mathbf{E}\rangle e^{(-\alpha-\epsilon)x}$$

$$\Rightarrow \langle \mathbf{U}(y), \mathbf{E}\rangle \equiv \langle \mathbf{V}(y), \mathbf{E}\rangle \equiv 0$$

proving lemma 13 2

Lemma 13.3

If $\|\mathbf{E}\|^2 = \epsilon^2 > 0$ *then we may assume*

$$f(x,y) = r_1 e^{-\alpha x} \begin{pmatrix} \frac{a_1}{r_1 \epsilon} \cos \alpha y \\ \cos \alpha y \cosh \epsilon y \sin(\epsilon x + r_2) + \sin \alpha y \sinh \epsilon y \sin(\epsilon x - r_2) \\ \cos \alpha y \sinh \epsilon y \sin(\epsilon x + r_2) + \sin \alpha y \cosh \epsilon y \sin(\epsilon x - r_2) \end{pmatrix}$$

where $r_1, r_2 \in \mathbb{R}$

Proof :

We shall first use the fact that

$$a \cos(\alpha y + t) = a_1 \cos \alpha y + a_2 \sin \alpha y$$

for some $a_1, a_2 \in \mathbb{R}$ and hence

$$\langle \mathbf{E}, f \rangle = (a_1 \cos \epsilon y + a_2 \sin \epsilon y) e^{-\alpha x}$$

recall from Lemma 13 2 that

$$f(x,y) = e^{-\alpha x}(\mathbf{U}(y)\cos(\epsilon x) + \mathbf{V}(y)\sin(\epsilon x) + \frac{1}{\epsilon^2}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E})$$

so that

$$\begin{aligned} f_1 \;=\; & \{ \cos \epsilon x \,[-\alpha \mathbf{U}(y) + \epsilon \mathbf{V}(y)] + \sin \epsilon x \,[-\alpha \mathbf{V}(y) - \epsilon \mathbf{U}(y)] \\ & -\frac{\alpha}{\epsilon^2}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E} \, \} \, e^{-\alpha x} \end{aligned}$$

$$f_2 = \left\{ \mathbf{U}'(y) \cos \epsilon x + \mathbf{V}'(y) \sin \epsilon x + \frac{\alpha}{\epsilon^2}(a_2 \cos \alpha y - a_1 \sin \alpha y)\mathbf{E} \right\} e^{-\alpha x}$$

and

$$
\begin{aligned}
f_{11} = \{ & \cos \epsilon x \left[\alpha^2 \mathbf{U}(y) - 2\alpha\epsilon \mathbf{V}(y) - \epsilon^2 \mathbf{U}(y)\right] \\
& + \sin \epsilon x \left[\alpha^2 \mathbf{V}(y) + 2\alpha\epsilon \mathbf{U}(y) - \epsilon^2 \mathbf{V}(y)\right] \\
& + \frac{\alpha^2}{\epsilon^2}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E} \ \} \, e^{-\alpha x}
\end{aligned}
$$

$$
f_{22} = \left\{ \mathbf{U}''(y) \cos \epsilon x + \mathbf{V}''(y) \sin \epsilon x - \frac{\alpha^2}{\epsilon^2}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E} \right\} e^{-\alpha x}
$$

Therefore the fact that

$$
f_{11} + f_{22} = 0 \Rightarrow \left\{ \begin{array}{rcl} \mathbf{U}''(y) + (\alpha^2 - \epsilon^2)\mathbf{U}(y) & = & 2\alpha\epsilon \mathbf{V}(y) \\ \\ \mathbf{V}''(y) + (\alpha^2 - \epsilon^2)\mathbf{V}(y) & = & -2\alpha\epsilon \mathbf{U}(y) \end{array} \right.
$$

which in turn imply

$$
\begin{aligned}
\mathbf{U}(y) &= e^{\epsilon y}(c_1 \cos \alpha y + c_3 \sin \alpha y) + e^{-\epsilon y}(c_2 \cos \alpha y + c_4 \sin \alpha y) \\
\mathbf{V}(y) &= e^{\epsilon y}(c_3 \cos \alpha y - c_1 \sin \alpha y) - e^{-\epsilon y}(c_4 \cos \alpha y - c_2 \sin \alpha y)
\end{aligned}
$$

for some constant vectors $c_1, c_2, c_3, c_4 \in \mathbb{R}^3$ The fact that $\langle f_1, f_2 \rangle = 0$ implies

$$
\begin{aligned}
\langle c_2, c_4 \rangle = \langle c_1, c_3 \rangle &= 0 \\
\langle c_3, c_3 \rangle &= \langle c_1, c_1 \rangle \\
\langle c_4, c_4 \rangle &= \langle c_2, c_2 \rangle
\end{aligned}
$$

$$\langle c_2, c_3\rangle + \langle c_1, c_4\rangle = \frac{\alpha^2}{\epsilon^2(\alpha^2+\epsilon^2)}(-a_1a_2)$$

$$\langle c_1, c_2\rangle + \langle c_3, c_4\rangle = \frac{\alpha^2}{\epsilon^2(\alpha^2+\epsilon^2)}(\frac{a_2^2-a_1^2}{2})$$

We now recall the fact that

$$\langle \mathbf{U}(y), \mathbf{E}\rangle = \langle \mathbf{V}(y), \mathbf{E}\rangle = 0$$

which implies

$$\langle c_i, \mathbf{E}\rangle = 0, \quad \forall i = 1,2,3,4$$

once these are satisfied we find all other conditions are automatically satisfied including the fact that $\langle f_1, f_1\rangle = \langle f_2, f_2\rangle$ So we have that

$$\langle c_1, \mathbf{E}\rangle = 0, \qquad \langle c_2, \mathbf{E}\rangle = 0$$

$$\langle c_1, c_1\rangle = \langle c_3, c_3\rangle \qquad \langle c_1, c_3\rangle = 0$$

As $\langle \mathbf{E}, \mathbf{E}\rangle = \epsilon^2 > 0$ from the preliminaries we know that $\mathbf{E}^\perp$ is a plane with metric $(-1,1)$ and hence $||c_i||^2, i = 1,2,3,4$ may be positive negative or zero Let us assume $||c_1||^2$ is positive, since $\langle c_3, \mathbf{E}\rangle = 0$ and $\langle c_3, c_1\rangle = 0$ we must have that $||c_3||^2$ is negative, but $\langle c_1, c_1\rangle = \langle c_3, c_3\rangle$ hence this is a contradiction A similar argument shows that $||c_1||^2$ cannot be negative Hence

$$\langle c_1, c_1\rangle = \langle c_3, c_3\rangle = 0 \tag{13 12}$$

as c_2 and c_4 have similar conditions imposed we also have

$$\langle c_2, c_2\rangle = \langle c_4, c_4\rangle = 0 \tag{13 13}$$

Thus

$$\langle c_i, \mathbf{E}\rangle = 0, \qquad \forall i = 1,2,3,4$$

$$\langle c_1, c_1\rangle = \langle c_3, c_3\rangle = \langle c_1, c_3\rangle = 0$$

$$\langle c_2, c_2\rangle = \langle c_4, c_4\rangle = \langle c_2, c_4\rangle = 0$$

$$\langle c_2, c_3\rangle + \langle c_1, c_4\rangle = \frac{\alpha^2}{\epsilon^2(\alpha^2+\epsilon^2)}(-a_1 a_2)$$

$$\langle c_1, c_2\rangle + \langle c_3, c_4\rangle = \frac{\alpha^2}{\epsilon^2(\alpha^2+\epsilon^2)}(\frac{a_2^2 - a_1^2}{2})$$

After a hyperbolic motion we may assume $\mathbf{E} = (\epsilon, 0, 0)^T$ and $f_y(0,0) = (s_1, 0, s_2)^T$, where $s_1, s_2 \in \mathbb{R}$ By switching to an associate we may also assume that $a_2 = 0$ and hence $f_y(0,0) = (0,0,s)$
As $\langle c_i, \mathbf{E}\rangle = 0$ and $\langle c_i, c_i\rangle = 0$ for each $i = 1,2,3,4$, we have

$$c_1 = (0, d_1, j_1 d_1)^T \qquad c_2 = (0, d_2, j_2 d_2)^T$$
$$c_3 = (0, d_3, j_3 d_3)^T \qquad c_4 = (0, d_4, j_4 d_4)^T$$

where $j_i = \pm 1, i = 1,2,3,4$ Given $\langle c_1, c_3\rangle = 0$ and $\langle c_2, c_4\rangle = 0$ we have $j_1 = j_3$ and $j_2 = j_4$ If $j_1 = j_2$ then f would be planar hence letting $j = j_1$ we have $j = j_1 = j_3 = -j_2 = -j_4$ Now $\langle c_2, c_3\rangle + \langle c_1, c_4\rangle = 0$ so

$$d_2 d_3 + d_1 d_4 = 0$$

and as $f_y(0,0) = (0,0,s)^T$ we also have

$$\epsilon d_2 = \epsilon d_1 - \alpha d_3 - \alpha d_4$$

hence either $d_4 = -d_3$ and $d_2 = d_1$ or $\alpha d_3 = -\epsilon d_1$ and $\epsilon d_2 = \alpha d_4$ The second of these conditions leads to f being planar Hence we now have

$$\begin{aligned} c_1 &= (0, d_1, j d_1)^T & c_2 &= (0, d_1, -j d_1)^T \\ c_3 &= (0, d_3, j d_3)^T & c_4 &= (0, -d_3, j d_3)^T \end{aligned}$$

and so

$$f(x,y) = \begin{pmatrix} \frac{a_1}{\epsilon} \cos \alpha y e^{-\alpha x} \\ \\ ((d_1 \cos \alpha y \cosh \epsilon y - d_3 \sin \alpha y \sinh \epsilon y) \cos \epsilon x \\ +(d_3 \cos \alpha y \cosh \epsilon y + d_1 \sin \alpha y \sinh \epsilon y) \sin \epsilon x) e^{-\alpha x} \\ \\ j((d_1 \cos \alpha y \sinh \epsilon y - d_3 \sin \alpha y \cosh \epsilon y) \cos \epsilon x \\ +(d_3 \cos \alpha y \sinh \epsilon y + d_1 \sin \alpha y \cosh \epsilon y) \sin \epsilon x) e^{-\alpha x} \end{pmatrix}$$

Checking that f now satisfies the original differential equation results in $j = 1$ With these conditions imposed we find all other condtions are automatically satisfied Letting $r_1 = \sqrt{d_1^2 + d_3^2}$ and $r_2 = \arctan(\frac{d_1}{d_3})$ we find these simplify to

$$f(x,y) = \begin{pmatrix} \frac{a_1}{\epsilon} \cos \alpha y e^{-\alpha x} \\ r_1 e^{-\alpha x} (\cos \alpha y \cosh \epsilon y \sin(\epsilon x + r_2) + \sin \alpha y \sinh \epsilon y \sin(\epsilon x - r_2)) \\ r_1 e^{-\alpha x} (\cos \alpha y \sinh \epsilon y \sin(\epsilon x + r_2) + \sin \alpha y \cosh \epsilon y \sin(\epsilon x - r_2)) \end{pmatrix}$$

and hence the lemma is proved

Lemma 13.4

If $\|\mathbf{E}\|^2 = -\epsilon^2 < 0, \epsilon^2 \neq \alpha^2$ *then we may assume*

$$f(x,y) = \begin{pmatrix} r_1 \cos(-\alpha+\epsilon)ye^{(-\alpha+\epsilon)x} + r_2 \cos(\alpha+\epsilon)ye^{(-\alpha-\epsilon)x} \\ -r_1 \sin(-\alpha+\epsilon)ye^{(-\alpha+\epsilon)x} - r_2 \sin(\alpha+\epsilon)ye^{(-\alpha-\epsilon)x} \\ -\frac{a_1}{\epsilon}e^{-\alpha x}\cos\alpha y \end{pmatrix}$$

where $r_1, r_2 \in \mathbb{R}$

Proof ·

From lemma 13 2 we have

$$f_{11} = \mathbf{U}(y)(-\alpha+\epsilon)^2 e^{(-\alpha+\epsilon)x} + \mathbf{V}(y)(-\alpha-\epsilon)^2 e^{(-\alpha-\epsilon)x} + \frac{a\alpha^2}{-\epsilon^2}e^{-\alpha x}\cos(\alpha y + t)\mathbf{E}$$

$$f_{22} = \mathbf{U}''(y)e^{(-\alpha+\epsilon)x} + \mathbf{V}''(y)e^{(-\alpha-\epsilon)x} - \frac{a\alpha^2}{-\epsilon^2}e^{-\alpha x}\cos(\alpha y + t)\mathbf{E}$$

Therefore

$$f_{11} + f_{22} = 0 \Rightarrow \left\{ \begin{aligned} \mathbf{U}''(y) + (-\alpha+\epsilon)^2\mathbf{U}(y) &= 0 \\ \mathbf{V}''(y) + (-\alpha-\epsilon)^2\mathbf{V}(y) &= 0 \end{aligned} \right.$$

Hence

$$\mathbf{U}(y) = v_1 \cos(-\alpha+\epsilon)y + v_2 \sin(-\alpha+\epsilon)y \tag{13 14}$$

$$\mathbf{V}(y) = v_3 \cos(\alpha+\epsilon)y + v_4 \sin(\alpha+\epsilon)y \tag{13 15}$$

for some $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ Also with the aid of the fact that

$$\langle \mathbf{U}(y), \mathbf{E}\rangle \equiv \langle \mathbf{V}(y), \mathbf{E}\rangle \equiv 0$$

$$\langle \mathbf{U}'(y), \mathbf{E}\rangle \equiv \langle \mathbf{V}'(y), \mathbf{E}\rangle \equiv 0$$

$$\langle E, v_i\rangle = 0 \quad i = 1, \ \ , 4$$

We now have

$$\begin{aligned} f(x,y) &= (v_1\cos(-\alpha+\epsilon)y + v_2\sin(-\alpha+\epsilon)y)e^{(-\alpha+\epsilon)x} \\ &\quad +(v_3\cos(\alpha+\epsilon)y + v_4\sin(\alpha+\epsilon)y)e^{(-\alpha-\epsilon)x} \\ &\quad -\frac{a}{\epsilon^2}e^{-\alpha x}\cos(\alpha y + t)\mathbf{E} \end{aligned}$$

$$\begin{aligned} f_1(x,y) &= (-\alpha+\epsilon)(v_1\cos(-\alpha+\epsilon)y + v_2\sin(-\alpha+\epsilon)y)e^{(-\alpha+\epsilon)x} \\ &\quad +(-\alpha-\epsilon)(v_3\cos(\alpha+\epsilon)y + v_4\sin(\alpha+\epsilon)y)e^{(-\alpha-\epsilon)x} \\ &\quad +\alpha\frac{a}{\epsilon^2}e^{-\alpha x}\cos(\alpha y + t)\mathbf{E} \end{aligned}$$

$$\begin{aligned} f_2(x,y) &= (-\alpha+\epsilon)(-v_1\sin(-\alpha+\epsilon)y + v_2\cos(-\alpha+\epsilon)y)e^{(-\alpha+\epsilon)x} \\ &\quad +(\alpha+\epsilon)(-v_3\sin(\alpha+\epsilon)y + v_4\cos(\alpha+\epsilon)y)e^{(-\alpha-\epsilon)x} \\ &\quad +\alpha\frac{a}{\epsilon^2}e^{-\alpha x}\sin(\alpha y + t)\mathbf{E} \end{aligned}$$

Using the fact that $\langle f_1, f_1\rangle = \langle f_2, f_2\rangle$ we have

$$(-\alpha+\epsilon)^2\langle \mathbf{U}(y), \mathbf{U}(y)\rangle = \langle \mathbf{U}'(y), \mathbf{U}'(y)\rangle \qquad (13\ 16)$$

$$(\alpha+\epsilon)^2\langle \mathbf{V}(y), \mathbf{V}(y)\rangle = \langle \mathbf{V}'(y), \mathbf{V}'(y)\rangle \qquad (13\ 17)$$

$$\langle \mathbf{U}(y), \mathbf{V}(y)\rangle(\alpha^2-\epsilon^2) - \langle \mathbf{U}'(y), \mathbf{V}'(y)\rangle = -\frac{a^2\alpha^2}{\epsilon^2}\cos 2(\alpha y + t) \qquad (13\ 18)$$

Substituting (13 14) and (13 15) into (13 16), (13 17) and (13 18) and evaluating at $y = 0$ results in

$$\begin{aligned} \langle c_1, c_1\rangle = \langle c_2, c_2\rangle & \qquad \langle c_3, c_3\rangle = \langle c_4, c_4\rangle \\ \langle c_1, c_3\rangle + \langle c_2, c_4\rangle &= \frac{\alpha^2}{2\epsilon^2(\alpha^2-\epsilon^2)}(a_2^2 - a_1^2) \end{aligned}$$

and on substituting these back we find

$$\begin{aligned} \langle c_1, c_2\rangle = 0 & \qquad \langle c_3, c_4\rangle = 0 \\ \langle c_1, c_4\rangle - \langle c_2, c_3\rangle &= \frac{\alpha^2}{2\epsilon^2(\alpha^2-\epsilon^2)}(2a_1a_2) \end{aligned}$$

After a hyperbolic motion we may assume $\mathbf{E} = (0,0,\epsilon)^T$ and $f_y(0,0) = (0,r_1,r_2)^T$ Then after switching to an associate we may assume $a_2 = 0$ and hence $f_y(0,0) = (0,r,0)^T$

As $\langle c_1, c_1\rangle = \langle c_2, c_2\rangle, \langle c_1, c_2\rangle = 0$ and $\langle \mathbf{E}, v_i\rangle = 0, i = 1,2$ we have

$$\begin{aligned} c_1 &= (r_1\cos t_1, r_1\sin t_1, 0)^T \\ c_2 &= (-j_1r_1\sin t_1, j_1r_1\cos t_1, 0)^T \end{aligned}$$

similarly

$$\begin{aligned} c_3 &= (r_2\cos t_2, r_2\sin t_2, 0)^T \\ c_4 &= (-j_2d_2\sin t_2, j_2r_2\cos t_2, 0)^T \end{aligned}$$

where $r_1, r_2, t_1, t_2 \in \mathbb{R}$ and $j_i = \pm 1, i = 1,2$ Now $\langle c_1, c_4\rangle = \langle c_2, c_3\rangle$ implying $-(j_1 + j_2)\sin(t1 - t2) = 0$ so that either $j_1 = -j_2$ or $j_1 = j_2$ and $t_2 = t_1 + n\pi$ for some integer n The first of these result in f being planar hence

$$\begin{aligned}
c_1 &= (r_1 \cos t_1, r_1 \sin t_1, 0)^T \\
c_2 &= (-j_1 r_1 \sin t_1, j_1 r_1 \cos t_1, 0)^T \\
c_3 &= (j_3 r_2 \cos t_1, j_3 r_2 \sin t_1, 0)^T \\
c_4 &= (-j_3 j_1 r_2 \sin t_1, j_3 j_1 r_2 \cos t_1, 0)^T
\end{aligned}$$

where $j_3 = \pm 1$ As $f_y(0,0) = (0, r, 0)^T$ we may assume $t_1 = n\pi/2$ where n is an integer Checking f satisfies the original differential equations result in $j_1 = -1$ By replacing $j_3 r_2$ with r_2 we may assume $j_3 = 1$ Finally replacing r_1 with $-r_1$ and r_2 with $-r_2$ if necessary we assume $t_1 = 0$ and hence

$$f(x,y) = \begin{pmatrix} r_1 \cos(-\alpha + \epsilon) y e^{(-\alpha+\epsilon)x} + r_2 \cos(\alpha + \epsilon) y e^{(-\alpha-\epsilon)x} \\ -r_1 \sin(-\alpha + \epsilon) y e^{(-\alpha+\epsilon)x} - r_2 \sin(\alpha + \epsilon) y e^{(-\alpha-\epsilon)x} \\ -\frac{a_1}{\epsilon} e^{-\alpha x} \cos \alpha y \end{pmatrix}$$

the lemma is proved

Lemma 13.5

If $\langle \mathbf{E}, \mathbf{E} \rangle = -\alpha^2$ *we may assume*

$$f(x,y) = \begin{pmatrix} r_2 e^{-2\alpha x}\cos(2\alpha y) - \frac{r_1}{2\alpha}x \\ -r_2 e^{-2\alpha x}\sin(2\alpha y) + \frac{r_1}{2\alpha}y \\ -\frac{a_1}{\alpha}e^{-\alpha x}\cos(\alpha y) \end{pmatrix}$$

where $r_1, r_2 \in \mathbb{R}$

Proof :

Recall

$$\begin{aligned} (13\ 3) \Rightarrow f_{11} &= -f_{22} \\ &= -(\alpha f_1 + \mathbf{E} * f_2) \\ &= -(2\alpha f_1 - \langle \mathbf{E}, f \rangle \mathbf{E} + c) \end{aligned}$$

therefore

$$f_{11} + 2\alpha f_1 = a\cos(\alpha y + t)e^{-\alpha x}\mathbf{E} - \mathbf{c} \tag{13 19}$$

To find a homogenous solution we look at

$$f_{11} = -2\alpha f_1$$

which may be written

$$f_1 = -2\alpha f + \mathbf{V}$$

which in turn has solution

$$f = \mathbf{U}e^{-2\alpha x} + \frac{\mathbf{V}}{2\alpha}$$

for a particular solution of (13 19) we try

$$P(x,y) = \mathbf{W}_1(y)a\cos(\alpha y + t)e^{-\alpha x}\mathbf{E} + cx\mathbf{W}_2$$

and substituting this in, we find

$$\mathbf{W}_1 = -1/\alpha^2 \quad \text{and} \quad \mathbf{W}_2 = -1/(2\alpha)$$

and hence

$$f(x,y) = \mathbf{U}(y)e^{-2\alpha x} + \frac{1}{2\alpha}\mathbf{V}(y) - \frac{a}{\alpha^2}\cos(\alpha y + t)e^{-\alpha x}\mathbf{E} - \frac{1}{2\alpha}cx$$

Now take the innerproduct with $\mathbf{E}$ across this equation to give

$$\langle \mathbf{U}(y), \mathbf{E}\rangle \equiv \langle \mathbf{V}(y), \mathbf{E}\rangle \equiv 0$$

Recalling $f_{11} + f_{22} = 0$ we have

$$f_{11} = 4\alpha^2\mathbf{U}(y)e^{-2\alpha x} - ae^{-\alpha x}\cos(\alpha y + t)\mathbf{E}$$

and

$$f_{22} = \mathbf{U}''(y)e^{-2\alpha x} + \frac{1}{2\alpha}\mathbf{V}''(y) + ae^{-\alpha x}\cos(\alpha y + t)\mathbf{E}$$

hence

$$\begin{aligned} \mathbf{V}''(y) &= 0 \\ \mathbf{U}''(y) + 4\alpha^2\mathbf{U}(y) &= 0 \end{aligned}$$

and so

$$\begin{aligned} \mathbf{V}(y) &= v_1 y + v_2 \\ \mathbf{U}(y) &= v_3\cos(2\alpha y) + v_4\sin(2\alpha y) \end{aligned}$$

for some $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ Also with the aid of the fact that

$$\langle \mathbf{U}(y), \mathbf{E}\rangle \equiv \langle \mathbf{V}(y), \mathbf{E}\rangle \equiv 0$$

$$\langle \mathbf{U}'(y), \mathbf{E}\rangle \equiv \langle \mathbf{V}'(y), \mathbf{E}\rangle \equiv 0$$

we get

$$\langle \mathbf{E}, v_i\rangle = 0 \quad i = 1, \ ,4$$

Now $a\cos(\alpha y + t) = a_1 \cos \alpha y + b \sin \alpha y$ for some $a_1, a_2 \in \mathbb{R}$ and we let $v_5 = -c$

$$\begin{aligned} f(x,y) &= (v_3 \cos(2\alpha y) + v_4 \sin(2\alpha y))e^{-2\alpha x} + \frac{1}{2\alpha}(v_5 x + v_1 y + v_2) \\ &\quad - \frac{1}{\alpha^2} e^{-\alpha x}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E} \end{aligned}$$

$$\begin{aligned} f_1(x,y) &= -2\alpha(v_3 \cos(2\alpha y) + v_4 \sin(2\alpha y))e^{-2\alpha x} + \frac{1}{2\alpha} v_5 \\ &\quad + \frac{1}{\alpha} e^{-\alpha x}(-a_1 \sin \alpha y - a_2 \cos \alpha y)\mathbf{E} \end{aligned}$$

$$\begin{aligned} f_2(x,y) &= -2\alpha(v_3 \cos(2\alpha y) - v_4 \sin(2\alpha y))e^{-2\alpha x} + \frac{1}{2\alpha} v_1 \\ &\quad + \frac{1}{\alpha} e^{-\alpha x}(-a_1 \sin \alpha + a_2 \cos \alpha y)\mathbf{E} \end{aligned}$$

after checking $\langle f_1, f_2 \rangle = 0$ we find

$$\begin{aligned} \langle v_3, v_3 \rangle &= \langle v_4, v_4 \rangle \\ \langle v_3, v_4 \rangle &= 0 \\ \langle v_5, v_1 \rangle &= 0 \\ \langle v_4, v_5 \rangle - \langle v_1, v_3 \rangle &= -a_1 a_2 \\ \langle v_3, v_5 \rangle + \langle v_1, v_4 \rangle &= \frac{a_2^2 - a_1^2}{2} \end{aligned}$$

we also check $\langle f_1, f_1 \rangle = \langle f_2, f_2 \rangle$ and so

$$\langle v_5, v_5 \rangle = \langle v_1, v_1 \rangle$$

hence

$$\begin{aligned} f(x,y) &= (v_3 \cos(2\alpha y) + v_4 \sin(2\alpha y))e^{-2\alpha x} + \frac{1}{2\alpha}(v_5 x + v_1 y + v_2) \\ &\quad - \frac{1}{\alpha^2} e^{-\alpha x}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E} \end{aligned}$$

and

$$\begin{aligned} \langle v_3, v_3\rangle = \langle v_4, v_4\rangle &\qquad \langle v_3, v_4\rangle = 0 \\ \langle v_5, v_5\rangle = \langle v_1, v_1\rangle &\qquad \langle v_5, v_1\rangle = 0 \\ \langle v_4, v_5\rangle - \langle v_1, v_3\rangle = -a_1a_2 &\qquad \langle v_3, v_5\rangle + \langle v_1, v_4\rangle = \frac{a_2^2 - a_1^2}{2} \\ \langle E, v_i\rangle = 0 &\qquad \forall i = 1,2,3,4,5 \end{aligned}$$

After a hyperbolic motion we may assume $\mathbf{E} = (0,0,s)^T$ and $f_y(0,0) = (0,js,0)^T$ for some $s \in \mathbb{R}, j = \pm 1$, By switching to an associate we may also assume $a_2 = 0$ Now if f is a solution to the original differential equations then so is $f + v$ where $v \in \mathbb{R}^3$ and hence we may assume $v_2 = (0,0,0)^T$ The conditions

$$\begin{aligned} \langle v_3, v_3\rangle = \langle v_4, v_4\rangle &\qquad \langle v_3, v_4\rangle = 0 \\ \langle v_5, v_5\rangle = \langle v_1, v_1\rangle &\qquad \langle v_5, v_1\rangle = 0 \\ \langle E, v_i\rangle = 0 &\qquad \forall i = 1,2,3,4,5 \end{aligned}$$

imply

$$\begin{aligned} v_1 &= (r_1 \cos t_1, r_1 \sin t_1, 0)^T \\ v_5 &= (-j_1 r_1 \sin t_1, j_1 r_1 \cos t_1, 0)^T \\ v_3 &= (r_2 \cos t_2, r_2 \sin t_2, 0)^T \\ v_4 &= (-j_2 r_2 \sin t_2, j_2 r_2 \cos t_2, 0)^T \end{aligned}$$

where $r_1, r_2, t_1, t_2 \in \mathbb{R}$ and $j_1, j_2 = \pm 1$

The conditions $\langle v_3, v_5\rangle = \langle v1, v_3\rangle$ and $\langle v_3, v_5\rangle + \langle v_1, v_4\rangle = 0$ together imply $j_3 = -j_1$ and $t_2 = t_1 + n\pi/2$ where n is an odd integer Hence we may assume

$$\begin{aligned}
v_1 &= (r_1 \cos t_1, r_1 \sin t_1, 0)^T \\
v_5 &= (-\jmath_1 r_1 \sin t_1, \jmath_1 r_1 \cos t_1, 0)^T \\
v_3 &= (\jmath_3 r_2 \sin t_2, \jmath_3 r_2 \cos t_2, 0)^T \\
v_4 &= (\jmath_3 \jmath_1 r_2 \cos t_2, -\jmath_3 \jmath_1 r_2 \sin t_2, 0)^T
\end{aligned}$$

where $\jmath_3 = \pm 1$ As we may assume the first coordinate of $f_1(0,0)$ is 0 we have $t_1 = \pm\pi/2$ By replacing $\jmath_3 r_2$ with r_2 we may assume $\jmath_3 = 1$ Checking that f now satisfies the original differential equations results in $\jmath_3 = 1$ Finally replacing r_1 with $-r_1$ and r_2 with $-r_2$ if necessary we may assume $t = \pi/2$ and hence

$$f(x,y) = \begin{pmatrix} r_2 e^{-2\alpha x} \cos(2\alpha y) - \frac{r_1}{2\alpha} x \\ -r_2 e^{-2\alpha x} \sin(2\alpha y) + \frac{r_1}{2\alpha} y \\ -\frac{a_1}{\alpha} e^{-\alpha x} \cos(\alpha y) \end{pmatrix}$$

proving the lemma

Lemma 13 6

If $||\mathbf{E}||^2 = 0$ *then after a translation in* $\mathbb{R}^3$ *we may assume*

$$f(x,y) = (\mathbf{U}(y) + x\mathbf{V}(y) + \frac{a}{2}x^2\cos(\alpha y + t)\mathbf{E})e^{-\alpha x}$$

where $\langle \mathbf{V}(y), \mathbf{E}\rangle = 0$ *and* $\langle \mathbf{U}(y), \mathbf{E}\rangle = a\cos(\alpha y + t)$

Proof :

$$\begin{aligned} f_{11} &= -f_{22} \\ &= -(\alpha f_1 + \mathbf{E} * f_2) \\ &= -(2\alpha f_1 + \alpha^2 f - \langle \mathbf{E}, f\rangle\mathbf{E}) \end{aligned}$$

therefore

$$f_{11} + 2\alpha f_1 + \alpha^2 f = \grave{a}\cos(\alpha y + t)e^{-\alpha x}\mathbf{E} \tag{13 20}$$

The homogenous equation

$$f_{11} + 2\alpha f_1 + \alpha^2 f = 0$$

has characteristic equation

$$\lambda^2 + 2\alpha\lambda + \alpha^2 = 0$$

which has a single root

$$\lambda = -\alpha$$

Thus we have

$$f(x,y) = (\mathbf{U}(y) + x\mathbf{V}(y))e^{-\alpha x} + (\text{a particular solution})$$

for a particular solution of (13 20) we try

$$P(x,y) = \mathbf{W}(y)\frac{x^2}{2}e^{-\alpha x}$$

and substituting in we get

$$\mathbf{W}(y)(1 - 2\alpha x + \alpha^2\frac{x^2}{2} + 2\alpha x - \alpha^2 x^2 + \alpha^2\frac{x^2}{2})e^{-\alpha x} = a\cos(\alpha y + t)e^{-\alpha x}\mathbf{E}$$

and simplifying we get

$$\mathbf{W}(y) = a\cos(\alpha y + t)\mathbf{E}$$

Thus

$$f(x,y) = (\mathbf{U}(y) + x\mathbf{V}(y) + \frac{a}{2}x^2\cos(\alpha y + t)\mathbf{E})e^{-\alpha x}$$

Now take the innerproduct with $\mathbf{E}$ across this equation to give

$$\langle \mathbf{E}, f\rangle = (\langle \mathbf{U}(y), \mathbf{E}\rangle + x\langle \mathbf{V}(y), \mathbf{E}\rangle + \frac{a}{2}\cos(\alpha y + t)\langle \mathbf{E}, \mathbf{E}\rangle)e^{-\alpha x}$$

from lemma 1 $\langle \mathbf{E}, f\rangle = ae^{-\alpha x}\cos(\alpha y + t)$

$$a\cos(\alpha y + t)e^{-\alpha x} = (\langle \mathbf{U}(y), \mathbf{E}\rangle + x\langle \mathbf{V}(y), \mathbf{E}\rangle)e^{-\alpha x}$$

$$\Rightarrow \langle \mathbf{U}(y), \mathbf{E}\rangle = a\cos(\alpha y + t) \text{ and } \langle \mathbf{V}(y), \mathbf{E}\rangle \equiv 0$$

proving the lemma

Lemma 13.7

If $\|\mathbf{E}\|^2 = 0$ *then we may assume*

$$f(x,y) = \frac{a_1 r_1}{2} e^{-\alpha x} \begin{pmatrix} [\frac{1}{r_1^2} + \frac{1}{\alpha^2} + (x+r_2)^2 - y^2] \cos \alpha y + 2y(x+r_2) \sin \alpha y \\ \frac{2}{r_1}(-y \cos \alpha y + (x+r_2) \sin \alpha y) \\ [\frac{-1}{r_1^2} + \frac{1}{\alpha^2} + (x+r_2)^2 - y^2] \cos \alpha y + 2y(x+r_2) \sin \alpha y \end{pmatrix}$$

for some $r_1, r_2 \in \mathbb{R}$

Proof :

We shall first use the fact that

$$a \cos(\alpha y + t) = a_1 \cos \alpha y + a_2 \sin \alpha y$$

for some $a_1, a_2 \in \mathbb{R}$ and hence

$$\langle \mathbf{E}, f \rangle = (a_1 \cos \epsilon y + a_2 \sin \epsilon y) e^{-\alpha x}$$

Now

$$f(x,y) = (\mathbf{U}(y) + x\mathbf{V}(y) + \frac{x^2}{2}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E}) e^{-\alpha x}$$

and hence

$$f_1 = (-\alpha \mathbf{U} + (1 - \alpha x)\mathbf{V} + (x - \frac{\alpha x^2}{2})(a_1 \cos \alpha y + a_2 \sin \alpha y)E) e^{-\alpha x}$$

$$f_2 = (\mathbf{U}'(y) + x\mathbf{V}'(y) - \frac{\alpha x^2}{2}(-a_2 \cos \alpha y + a_1 \sin \alpha y)\mathbf{E}) e^{-\alpha x}$$

and

$$
\begin{aligned}
f_{11} &= (\alpha^2\mathbf{U} + (\alpha^2 x - 2\alpha)\mathbf{V} + (\frac{\alpha^2 x^2}{2} - 2\alpha x + 1)(a_1 \cos \alpha y + a_2 \sin \alpha y)E)e^{-\alpha x} \\
f_{22} &= (\mathbf{U}''(y) + x\mathbf{V}''(y) - \frac{\alpha^2 x^2}{2}(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E})e^{-\alpha x}
\end{aligned}
$$

Since $f_{11} + f_{22} = 0$ we have

$$
(\mathbf{U}''(y) + \alpha^2\mathbf{U} - 2\alpha\mathbf{V}) + (x\mathbf{V}''(y) + \alpha^2 x\mathbf{V}) = (2\alpha x - 1)(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E}
$$

therefore

$$
\begin{aligned}
\mathbf{V}''(y) + \alpha^2\mathbf{V}(y) &= 2\alpha(a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E} \\
\mathbf{U}''(y) + \alpha^2\mathbf{U}(y) &= 2\alpha\mathbf{V}(y) - (a_1 \cos \alpha y + a_2 \sin \alpha y)\mathbf{E}
\end{aligned}
$$

implying

$$
\begin{aligned}
\mathbf{V}(y) &= (c_1 + \frac{1}{2\alpha}(a_1(\cos 2\alpha y - 1) + a_2 \sin 2\alpha y)\mathbf{E}) \cos \alpha y \\
&\quad + (c_2 + \frac{1}{2\alpha}(a_2(1 - \cos 2\alpha y) + a_1 \sin 2\alpha y)\mathbf{E}) \sin \alpha y \\
&\quad + y\left[-a_2 \cos \alpha y + a_1 \sin \alpha y\right]\mathbf{E}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{U}(y) &= (c_3 + \frac{1}{2\alpha}(c_1(\cos 2\alpha y - 1) + (c_2 + \frac{a_2}{\alpha}\mathbf{E})\sin 2\alpha y))\cos\alpha y \\
&+(c_4 + \frac{1}{2\alpha}(c_2 + \frac{a_2}{\alpha}\mathbf{E} - (c_2 + \frac{a_2}{\alpha}\mathbf{E})\cos 2\alpha y + c_1\sin 2\alpha y)\sin\alpha y \\
&+y\left[(-c_2 + \frac{1}{2\alpha}(a_1\sin 2\alpha y - a_2(1+\cos 2\alpha y))\mathbf{E})\cos\alpha y\right. \\
&\left.+(c_1 + \frac{1}{2\alpha}(-a_1(1+\cos 2\alpha y) - a_2\sin 2\alpha y)\mathbf{E})\sin\alpha y\right] \\
&-\frac{y^2}{2}\left[a_1\cos\alpha y + a_2\sin\alpha y\right]\mathbf{E}
\end{aligned}
$$

for some $c_1, c_2, c_3, c_4 \in \mathbb{R}^3$ As

$$
\begin{aligned}
\langle \mathbf{E}, \mathbf{U}(y)\rangle &= a_1\cos\epsilon y + a_2\sin\epsilon y \\
\langle \mathbf{E}, \mathbf{V}(y)\rangle &= 0
\end{aligned}
$$

we see that

$$
\langle \mathbf{E}, c_1\rangle = 0 \qquad \langle \mathbf{E}, c_2\rangle = 0
$$

$$
\langle \mathbf{E}, c_3\rangle = a_1 \qquad \langle \mathbf{E}, c_4\rangle = a_2
$$

The condition $\langle f_1, f_2\rangle = 0$ results in the following equalities

$$
\langle c_1, c_2\rangle = -a_1 a_2
$$

$$
\begin{aligned}
\langle c_2, c_2\rangle &= \langle c_1, c_1\rangle + a_1^2 - a_2^2 \\
\langle c_1, c_4\rangle &= -\langle c_2, c_3\rangle
\end{aligned}
$$

$$\begin{aligned}\langle c_2, c_4\rangle &= \langle c_1, c_3\rangle - \frac{1}{\alpha}(a_1^2 + \langle c_1, c_1\rangle)\\ \langle c_3, c_4\rangle &= -\frac{1}{\alpha}\langle c_2, c_3\rangle\\ \langle c_4, c_4\rangle &= \frac{1}{\alpha^2}\langle c_1, c_1\rangle - \frac{2}{\alpha}\langle c_1, c_3\rangle + \langle c_3, c_3\rangle\end{aligned}$$

these also satisfy the condition $\langle f_1, f_1\rangle = \langle f_2, f_2\rangle$ and so we have

$$\begin{aligned}f(x,y) &= (c_3 + \frac{1}{2\alpha}(c_1(\cos 2\alpha y - 1) + (c_2 + \frac{a_2}{\alpha}\mathbf{E})\sin 2\alpha y))\cos\alpha y\\ &+(c_4 + \frac{1}{2\alpha}(c_2 + \frac{a_2}{\alpha}\mathbf{E} - (c_2 + \frac{a_2}{\alpha}\mathbf{E})\cos 2\alpha y + c_1\sin 2\alpha y)\sin\alpha y\\ &+x\left[(c_1 + \frac{1}{2\alpha}(a_1(\cos 2\alpha y - 1) + a_2\sin 2\alpha y)\mathbf{E})\cos\alpha y\right.\\ &\left.+(c_2 + \frac{1}{2\alpha}(a_2(1 - \cos 2\alpha y) + a_1\sin 2\alpha y)\mathbf{E})\sin\alpha y\right]\\ &+y\left[(-c_2 + \frac{1}{2\alpha}(a_1\sin 2\alpha y - a_2(1 + \cos 2\alpha y))\mathbf{E})\cos\alpha y\right.\\ &\left.+(c_1 + \frac{1}{2\alpha}(-a_1(1 + \cos 2\alpha y) - a_2\sin 2\alpha y)\mathbf{E})\sin\alpha y\right]\\ &\{\frac{x^2 - y^2}{2}(a_1\cos\alpha y + a_2\sin\alpha y) + xy(-a_2\cos\alpha y + a_1\sin\alpha y)\}\mathbf{E}\ ]\, e^{-\alpha x}\end{aligned}$$

for some constants $a_1, a_2 \in \mathbb{R}$ and some constant vectors $c_1, c_2, c_3, c_4 \in \mathbb{R}^3$ with the following conditions

$$\langle \mathbf{E}, c_1 \rangle = 0 \qquad \langle \mathbf{E}, c_2 \rangle = 0$$

$$\langle c_1, c_2 \rangle = -a_1 a_2 \qquad \langle c_2, c_2 \rangle = \langle c_1, c_1 \rangle + a_1^2 - a_2^2$$

$$\langle \mathbf{E}, c_3 \rangle = a_1, \qquad \langle \mathbf{E}, c_4 \rangle = a_2$$

$$\langle c_1, c_4 \rangle = -\langle c_2, c_3 \rangle \qquad \langle c_3, c_4 \rangle = -\frac{1}{\alpha}\langle c_2, c_3 \rangle$$

$$\langle c_2, c_4 \rangle = \langle c_1, c_3 \rangle - \frac{1}{\alpha}(a_1^2 + \langle c_1, c_1 \rangle)$$

$$\langle c_4, c_4 \rangle = \frac{1}{\alpha^2}\langle c_1, c_1 \rangle - \frac{2}{\alpha}\langle c_1, c_3 \rangle + \langle c_3, c_3 \rangle$$

After a hyperbolic motion we may assume the third component of $f_y(0,0)$ is zero Then after a further hyperbolic motion we may also assume $\mathbf{E} = (r_1, 0, r_1)^T$ for some $r \in \mathbb{R}$ Using (13 7) we have that the first component of $f_y(0,0)$ must also be zero and hence we may assume

$$\mathbf{E} = (r_1, 0, r_1)^T \quad \text{and} \quad f_y(0,0) = (0, b, 0)^T$$

for some $r_1, b \in \mathbb{R}$ Now $\langle f_x(0,0), f_y(0,0) \rangle = 0$ and so the second component of $f_x(0,0)$ is zero Also from (13 8) we have $\langle \mathbf{E}, f_x(0,0) \rangle = -\alpha a$ and hence

$$f_x(0,0) = (c, 0, c + \frac{\alpha a}{r_1})^T$$

for some $c \in \mathbb{R}$ Finally $\langle f_x(0,0), f_x(0,0) \rangle = \langle f_y(0,0), f_y(0,0) \rangle$ and hence $c = \frac{r_1}{2\alpha a}(-b^2 - \frac{\alpha^2 a^2}{r_1^2})$ and so

$$f_x(0,0) = (-\frac{b}{2}(\frac{r_1 b}{\alpha a} + \frac{\alpha a}{r_1 b}), 0, -\frac{b}{2}(\frac{r_1 b}{\alpha a} - \frac{\alpha a}{r_1 b}))^T$$

From above we know the value of $\langle \mathbf{E}, c_i \rangle$ for $i = 1, \ldots, 4$ and given $\mathbf{E} = (r_1, 0, r_1)^T$ we have

$$
\begin{aligned}
c_1 &= (d_1, d_2, d_1)^T \\
c_2 &= (d_3, d_4, d_3)^T \\
c_3 &= (d_5 + a_1/r_1, d_6, d_5)^T \\
c_4 &= (d_7 + a_2/r_1, d_8, d_7)^T
\end{aligned}
$$

Since $\langle c_1, c_2 \rangle = 0$ then $d_2 d_4 = 0$ i.e. either $d_2 = 0$ or $d_4 = 0$. Also $\langle c_2, c_2 \rangle = \langle c_1, c_1 \rangle + a_1^2 - a_2^2$ hence $d_4^2 = d_2^2 + a_1^2$. Now if $d_4 = 0$ we would have to have $a_1 = 0$ making f planar thus $d_2 = 0$ and $d_4 = j a_1$ where $j = \pm 1$. Checking $\langle c_1, c_4 \rangle + \langle c_2, c_3 \rangle = 0$ leads to $d_3 = -j r_1 d_6$, where $j = \pm 1$ and the condition $\langle c_1, c_4 \rangle + \frac{1}{\alpha} \langle c_2, c_3 \rangle = 0$ implies $d_7 = -r_1/a_1 d_6 d_8$. Given $\langle c_2, c_4 \rangle = \langle c_1, c_3 \rangle - \frac{1}{\alpha}(a_1^2 + \langle c_1, c_1 \rangle)$ we have $d_1 = a_1 r_1/\alpha + d_8 j r_1$. Finally checking $\langle c_4, c_4 \rangle + 2/\alpha \langle c_1, c_3 \rangle = \langle c_3, c_3 \rangle + 1/\alpha^2 \langle c_1, c_1 \rangle$ results in $d_5 = r_1/(2a_1)(2a_1^2/a^2 - d_6^2 + d_8^2 + 2a_1 d_8 j/\alpha - a_1^2/r_1^2)$. Now as

$$
f_x(0,0) = \left(-\frac{b}{2}\left(\frac{r_1 b}{\alpha a} + \frac{\alpha a}{r_1 b}\right), 0, -\frac{b}{2}\left(\frac{r_1 b}{\alpha a} - \frac{\alpha a}{r_1 b}\right)\right)^T
$$

we must have $d_6 = 0$ and since $f_y(0,0) = (0, b, 0)^T$ we also have $d_8 = b/\alpha$. Checking that f satisfies the original differential equations implies $j = 1$. Thus on simplifying we have

$$
f(x,y) = \frac{a_1 r_1}{2} e^{-\alpha x}
\begin{pmatrix}
[\frac{1}{r_1^2} + \frac{1}{\alpha^2} + (x + r_2)^2 - y^2] \cos \alpha y + 2y(x + r_2) \sin \alpha y \\
\frac{2}{r_1}(-y \cos \alpha y + (x + r_2) \sin \alpha y) \\
[\frac{-1}{r_1^2} + \frac{1}{\alpha^2} + (x + r_2)^2 - y^2] \cos \alpha y + 2y(x + r_2) \sin \alpha y
\end{pmatrix}
$$

where $r_2 = \frac{b+\alpha}{a_1 \alpha}$ and so the lemma is proved

We now study the *Minimal* Surfaces when $\alpha = 0$

The differential equations (9 20) and (9 22) in this case reduce to the form

$$f_2 = E * f \tag{13 21}$$

$$f_{11} + f_{22} = 0 \tag{13 22}$$

We note that $\tilde{f} = f + sE, s \in \mathbb{R}$ satisfies the differential equation once f does Now (13 21) and (13 22) imply

$$\langle E, f_2 \rangle = \langle E, f_{11} \rangle = 0$$

and so

$$\langle E, f \rangle = -c(x + d)$$

for some $c, d \in \mathbb{R}$

$$\begin{aligned} f_{22} &= E * (E * f) \\ &= \langle E, E \rangle f - \langle E, f \rangle E \\ &= \langle E, E \rangle f + c(x + d) E \end{aligned}$$

and so

$$f_{22} - \|E\|^2 f = c(x + d) E \tag{13 23}$$

At this point we shall split the analysis into three cases

- $\langle E, E \rangle = 0$
- $\langle E, E \rangle < 0$
- $\langle E, E \rangle > 0$

- $\langle E, E\rangle = 0$

$$
\begin{aligned}
f_{22} &= c(x+d)E \\
f_2 &= c(xy+dy)E + g_1(x) \quad \text{where } g_1(x) \in \mathbb{R}^3 \\
f &= \frac{c}{2}(xy^2+dy^2)E + g_1(x)y + g_2(x) \quad \text{where } g_2(x) \in \mathbb{R}^3 \\
f_1 &= \frac{c}{2}y^2E + g_1'(x)y + g_2'(x) \\
f_{11} &= g_1''(x)y + g_2''(x) \\
&= -f_{22} = -c(x+d)E
\end{aligned}
$$

Hence $g_1''(x) = 0$ and $g_2''(x) = -c(x+d)E$, i e

$$g_1(x) = v_2x + v_3$$

and

$$g_2(x) = -c(\frac{1}{6}x^3 + \frac{1}{2}dx^2)E + v_4x + v_5$$

resulting in

$$f = \frac{c}{6}(-x^3 + 3xy^2 + 3dy^2 - 3dx^2)E + v_2xy + v_3y + v_4x + v_5$$

and so

$$
\begin{aligned}
E * f &= E * v_2xy + E * v_3y + E * v_4x + E * v_5 \\
&= f_2 \\
&= cxyE + cdyE + v_2x + v_3
\end{aligned}
$$

giving

$$\begin{aligned} E * v_2 &= cE \\ E * v_3 &= cdE \\ E * v_4 &= v_2 \\ E * v_5 &= v_3 \end{aligned}$$

Now after a hyperbolic rotation and a stretching we may assume that $E = (r, 0, r)^T$ and hence we have

$$\begin{aligned} v_2 &= (s_2, c, s_2)^T \\ v_3 &= (s_3, cd, s_3)^T \\ v_4 &= (s_4, \frac{s_2}{r}, s_4 + \frac{c}{r})^T \\ v_5 &= (s_5, \frac{s_3}{r}, s_5 + \frac{cd}{r})^T \end{aligned}$$

where $s_2, s_3, s_4, s_5 \in \mathbb{R}$ We may also assume $f_y(0,0) = (0, -bl, 0)^T$ and $f_x(0,0) = ((b^2r^2+1)l/(2r), 0, (b^2r^2-1)l/(2r))^T$ for some $b, l \in \mathbb{R}$ and hence

$$\begin{aligned} c &= -l \\ d &= b \\ s_3 &= 0 \\ s_2 &= 0 \\ s_4 &= lb/2(rb + 1/(rb)) \end{aligned}$$

resulting in

$$f(x,y) = \frac{lr}{6}(x+b)\begin{pmatrix} 3/r^2 + (x+b)^2 - 3y^2 \\ -6/ry \\ -3/r^2 + (x+b)^2 - 3y^2 \end{pmatrix} + \begin{pmatrix} s_6 + cd/(2r) + cd^3r/6 \\ 0 \\ s_6 + cd/(2r) + cd^3r/6 \end{pmatrix}$$

for some $s_6 \in \mathbb{R}$

It is worth noting that $\langle f_y, f_y \rangle = \langle f_x, f_x \rangle$ and $\langle f_x, f_y \rangle = 0$ for all $x, y \in \mathbb{R}$ and so f satisfies all the conditions for minimality

Recall that if f is a solution of the differential equations then so is $\tilde{f} = f + sE$ hence we may assume $s_6 = -cd/(2r) - cd^3r/6$ and hence

$$f(x,y) = \frac{lr}{6}(x+b)\begin{pmatrix} 3/r^2 + (x+b)^2 - 3y^2 \\ -6/ry \\ -3/r^2 + (x+b)^2 - 3y^2 \end{pmatrix} \qquad (13\ 24)$$

This is Enneper's surface of the second kind, which is a minimal spacelike surface of revolution

- $\langle E, E\rangle = e^2 > 0$

Hence we need to solve

$$f_{22} - e^2 f = c(x+d)E$$

Solution of Homogenous Equation

$$g_1(x,y) = v_2(x)\sinh ey + v_3(x)\cosh ey$$

for some $v_2(x), v_3(x) \in \mathbb{R}^3$ A particular solution is

$$g_2(x,y) = -\frac{c}{e^2}(x+d)E$$

Therefore the solution of this equation is

$$\begin{aligned} f &= v_2(x)\sinh ey + v_3(x)\cosh ey - ce^{-2}(x+d)E \\ E * f &= \sinh eyE * v_2(x) + \cosh eyE * v_3(x) \\ &= f_2 \\ &= ev_2(x)\cosh ey + ev_3(x)\sinh ey \end{aligned}$$

hence $ev_3(x) = E * v_2(x)$ and $ev_2(x) = E * v_3(x)$

$$\begin{aligned} -f_{22} &= -e^2 v_2(x)\sinh ey - e^2 v_3(x)\cosh ey \\ f_{11} &= v_2''(x)\sinh ey + v_3''(x)\cosh ey \end{aligned}$$

hence

$$\begin{aligned} v_2(x) &= v_4\cos ex + v_5\sin ex \\ v_3(x) &= v_6\cos ex + v_7\sin ex \end{aligned}$$

where $v_4, v_5, v_6, v_7 \in \mathbb{R}^3$

Hence

$$f = (v_4 \cos ex + v_5 \sin ex)\sinh ey + (v_6 \cos ex + v_7 \sin ex)\cosh ey - ce^{-2}(x+d)E$$

and

$$\begin{aligned} E * v_4 &= ev_6 \\ E * v_5 &= ev_7 \\ E * v_6 &= ev_4 \\ E * v_7 &= ev_5 \end{aligned}$$

Now after a hyperbolic motion and a strecthing we may assume $E = (0, \epsilon, 0)^T$ and hence

$$\begin{aligned} v_4 &= (s_1, 0, s_2)^T \\ v_5 &= (s_3, 0, s_4)^T \\ v_6 &= (-s_2, 0, -s_1)^T \\ v_7 &= (-s_4, 0, -s_3)^T \end{aligned}$$

and we may also assume $f_y(0,0) = (r\cos b, 0, 0)^T$ and $f_x(0,0) = (0, r, r\sin b)^T$ resulting in

$$\begin{array}{cc} s_4 = 0 & s_2 = 0 \\ s_1 = \dfrac{r}{e}\cos b & s_3 = -\dfrac{r}{e}\sin b \end{array}$$

and $c = -\frac{r}{\epsilon}\epsilon$, hence

$$f(x,y) = r\begin{pmatrix} \sinh(-\epsilon y)\cos(\epsilon x + \tan b) \\ \epsilon(x+d) \\ \cosh(-\epsilon y)\cos(\epsilon x + \tan b) \end{pmatrix}$$

and recalling that if f is a solution of the differential equation then so is $\tilde{f} = f + sE$ allows us assume $d = \tan b$ i e

$$f(x,y) = r_1 \begin{pmatrix} \sinh(-\epsilon y)\cos(\epsilon x + r_2) \\ \epsilon(x + r_2) \\ \cosh(-\epsilon y)\cos(\epsilon x + r_2) \end{pmatrix} \tag{13 25}$$

where $r_1 = \frac{r}{\epsilon}$ and $r_2 = \tan b$ This surface is the catenoid of the 2^{nd} kind, which is a minimal surface of revolution

- $\langle E, E\rangle = -e^2 < 0$

Hence we need to solve

$$f_{22} + e^2 f = c(x+d)$$

Solution of Homogenous Equation

$$g_1(x,y) = v_2(x)\sin ey + v_3(x)\cos ey$$

Particular Solution

$$g_2(x,y) = \frac{c}{e^2}(x+d)E$$

Therefore the solution of this equation is

$$\begin{aligned} f &= v_2(x)\sin ey + v_3(x)\cos ey + ce^{-2}(x+d)E \\ E * f &= \sin eyE * v_2(x) + \cos eyE * v_3(x) \\ &= f_2 \\ &= ev_2(x)\cos ey - ev_3(x)\sin ey \end{aligned}$$

hence $ev_2(x) = E * v_3(x)$ and $-ev_3(x) = E * v_2(x)$

$$\begin{aligned} -f_{22} &= e^2 v_2(x)\sin ey + e^2 v_3(x)\cos ey \\ f_{11} &= v_2''(x)\sin ey + v_3''(x)\cos ey \end{aligned}$$

Hence

$$\begin{aligned} v_2(x) &= v_4 \sinh ex + v_5 \cosh ex \\ v_3(x) &= v_6 \sinh ex + v_7 \cosh ex \end{aligned}$$

for some $v_4, v_5, v_6, v_7 \in \mathbb{R}^3$

Recall $ev_2(x) = E * v_3(x)$ and $-ev_3(x) = E * v_2(x)$, hence

$$f = (v_4 \sinh ex + v_5 \cosh ex) \sin ey + (v_6 \sinh ex + v_7 \cosh ex) \cos ey + ce^{-2}(x + d)E$$

with

$$\begin{aligned} ev_4 &= E * v_6 && (13\ 26) \\ ev_5 &= E * v_7 && (13\ 27) \\ ev_6 &= -E * v_4 && (13\ 28) \\ ev_7 &= -E * v_5 && (13\ 29) \end{aligned}$$

Now after a hyperbolic motion we may assume $E = (0, 0, \epsilon)^T$ hence

$$\begin{aligned} v_4 &= (s_1, -s_2, 0)^T \\ v_5 &= (s_3, -s_4, 0)^T \\ v_6 &= (s_2, s_1, 0)^T \\ v_7 &= (s_4, s_3, 0)^T \end{aligned}$$

we may also assume $f_x(0,0) = (r \cosh b, 0, -r)^T$ and $f_y(0,0) = (0, -r \sinh b, 0)^T$ and hence

$$f(x,y) = r \begin{pmatrix} \sinh(\epsilon x + b) \cos \epsilon y \\ -\sinh(\epsilon x + b) \sin \epsilon y \\ -\epsilon x + d \end{pmatrix}$$

finally recalling that if f is a solution of the differential equation then so is $\tilde{f} = f + sE$ lets us assume that $d = -b$ and so

$$f(x,y) = -r \begin{pmatrix} \sinh(\epsilon x + b) \cos y \\ -\sinh(\epsilon x + b) \sin y \\ -(\epsilon x + b) \end{pmatrix} \qquad (13\ 30)$$

this surface is the catenoid of the 1^{st} kind which is a minimal surface of revolution

Appendix A

Determination of $\nabla_{X_p} Y$

Proof that given g(,) so that equations (A 1) and (A 2) (given below) hold for all $X_p \in T_pM$ and for all smooth vector fields Y, that $\nabla_{X_p} Y$ is completely determined

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{A 1}$$

$$\nabla_{X_p} Y - \nabla_{Y_p} X = [X,Y]_p \tag{A 2}$$

from (A 1)

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \tag{A 3}$$

$$Yg(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \tag{A 4}$$

$$Zg(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \tag{A 5}$$

If we examine (A 3) + (A 4) - (A 5) we see

$$\begin{aligned}
Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) &= g(\nabla_X Y, Z) + g(\nabla_X Z - \nabla_Z X, Y) \\
&\quad + g(\nabla_Y X, Z) + g(\nabla_Y Z - \nabla_Z Y, X) \\
&= g(\nabla_X Y, Z) + g([X,Z],Y) \\
&\quad + g(\nabla_X Y - [X,Y], Z) + g([Y,Z],X) \\
&= 2g(\nabla_X Y, Z) + g([X,Z],Y) \\
&\quad - g([X,Y],Z) + g([Y,Z],X)
\end{aligned}$$

Therefore

$$\begin{aligned}
2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\
&\quad + g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X) \qquad \text{(A 6)}
\end{aligned}$$

The right hand side of (A 6) in independent of ∇ Suppose Z_1, Z_2 form orthonormal basis for T_pM then

$$\nabla_X Y = g(\nabla_X Y, Z_1)Z_1 + g(\nabla_X Y, Z_2)Z_2$$

and the right hand side of this equation is determined from above

Using the Gauss-Weingarten equations we can arrive at the following results

1 $g_p(R(X_p, Y_p)Y_p, X_p) = -det(A_p)$ where $R(X,Y)Z$ is the cuvature tensor, defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

2 $(\nabla_{X_p} A)Y = (\nabla_{Y_p} A)X$ called Codazzi's equation

Proof:

For all smooth vector fields X,Y,Z we have,

$$Y(Zf) = (f_*)(\nabla_Y Z) - g(AY, Z)\xi$$

$$\begin{aligned} X(Y(Zf)) &= X[(\nabla_Y Z)f] - [Xg(AY, Z)]\xi - g(AY, Z)X\xi \\ &= (f_*)(\nabla_X \nabla_Y Z) - g(AX, \nabla_Y Z)\xi \\ &\quad -[Xg(AY, Z)]\xi + g(AY, Z)(f_*)(AX) \end{aligned}$$

$$\begin{aligned} XY(Zf) &= (f_*)(\nabla_X \nabla_Y Z + g(AY, Z)(AX)) \\ &\quad -[g(AX, \nabla_Y Z) + Xg(AY, Z)]\xi \end{aligned} \tag{A 7}$$

$$\begin{aligned} YX(Zf) &= (f_*)(\nabla_Y \nabla_X Z + g(AX, Z)(AY)) \\ &\quad -[g(AY, \nabla_X Z) + Yg(AX, Z)]\xi \end{aligned} \tag{A 8}$$

$$[X, Y](Zf) = (f_*)(\nabla_{[X,Y]} Z) - g(A[X, Y], Z)\xi \tag{A 9}$$

but $XY - YX = [X, Y]$ thus (A 7) - (A 8) = (A 9) or 0 = (A 7) - (A 8) - (A 9) i e

$$\begin{aligned} 0 = {} & (f_*)[\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z + g(AY, Z)(AX) - g(AX, Z)(AY)] \\ & +[-g(AX, \nabla_Y Z) - Xg(AY, Z) \\ & \qquad + g(AY, \nabla_X Z) + Yg(AX, Z) + g(A[X, Y], Z)]\xi \end{aligned}$$

Thus,

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = g(AX, Z)(AY) - g(AY, Z)(AX)$$

and

$$g(AX, \nabla_Y Z) - g(AY, \nabla_X Z) + Xg(AY, Z) - Yg(AX, Z) - g(A[X,Y], Z) = 0$$

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, is called the curvature tensor So we have

$$R(X,Y)Z = g(AX, Z)(AY) - g(AY, Z)(AX)$$

Suppose X, Y are smooth vector fields on M such that X_p, Y_p form an orthonormal basis for T_pM with respect to $g_p(,)$, then

$$\begin{aligned} g(R(X,Y)Y, X) &= g([g(AX,Y)(AY) - g(AY,Y)(AX)], X) \\ &= g(AX,Y)g(AY,X) - g(AY,Y)g(AX,X) \\ &= -det\begin{pmatrix} g(AX,X) & g(AY,X) \\ g(AX,Y) & g(AY,Y) \end{pmatrix} \\ &= -det(\text{matrix representation of the linear map } \mathrm{A_p}) \\ &= -det A_p \end{aligned}$$

From the normal component we have

$$\begin{aligned} 0 &= g(AX, \nabla_Y Z) - g(\nabla_X Y - \nabla_Y X, Z) + g(\nabla_X(AY), Z) \\ &\quad + g(AY, \nabla_X Z) - g(\nabla_Y AX, Z) - g(AX, \nabla_Y Z) \end{aligned}$$

Therefore,

$g(\nabla_X(AY) - A(\nabla_X Y), Z) = g(\nabla_Y(AX) - A(\nabla_Y X), Z)$for all smooth vector fields Z

and so,

$$\nabla_X(AY) - A(\nabla_X Y) = \nabla_Y(AX) - A(\nabla_Y X)$$

Define,

$$(\nabla_{X_p} A)Y = \nabla_{X_p}(AY) - A(\nabla_{X_p} Y)$$

then we have

$$(\nabla_{X_p} A)Y = (\nabla_{Y_p} A)X$$

Appendix B

The Christoffel symbols

In the case when M has locally defined isothermal coordinates x_1, x_2 i e

$$g(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_1}) = g(\tfrac{\partial}{\partial x_2}, \tfrac{\partial}{\partial x_2}) = e^{\phi}$$

$$g(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}) = 0$$

Then

$$\Gamma^1_{11} = \tfrac{1}{2}\phi_x, \quad \Gamma^1_{22} = -\tfrac{1}{2}\phi_x, \quad \Gamma^1_{12} = \tfrac{1}{2}\phi_y, \quad \Gamma^1_{21} = \tfrac{1}{2}\phi_y$$

$$\Gamma^2_{11} = -\tfrac{1}{2}\phi_y, \quad \Gamma^2_{22} = \tfrac{1}{2}\phi_y, \quad \Gamma^2_{12} = \tfrac{1}{2}\phi_x, \quad \Gamma^2_{21} = \tfrac{1}{2}\phi_x$$

Proof .

$$\frac{\partial}{\partial y} g(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \phi_y e^{\phi}$$

$$2g(\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}, \frac{\partial}{\partial x}) = \phi_y e^{\phi}$$

$$2g(\Gamma^1_{21}\frac{\partial}{\partial x} + \Gamma^2_{21}\frac{\partial}{\partial y}, \frac{\partial}{\partial x}) = \phi_y e^{\phi}$$

$$2\Gamma^1_{21} e^{\phi} = \phi_y e^{\phi}$$

$$\Gamma^1_{21} = \frac{1}{2}\phi_y \tag{B 1}$$

Similarly

$$g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = e^{\phi}$$

$$\frac{\partial}{\partial x} g(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \phi_x e^{\phi}$$

$$2g(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \phi_x e^{\phi}$$

$$2g(\Gamma^1_{12}\frac{\partial}{\partial x} + \Gamma^2_{12}\frac{\partial}{\partial y}, \frac{\partial}{\partial y}) = \phi_x e^{\phi}$$

$$2\Gamma^2_{12} e^{\phi} = \phi_x e^{\phi}$$

$$\Gamma^2_{12} = \frac{1}{2}\phi_x \tag{B 2}$$

and

$$[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} - \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = 0$$

$$\Gamma^1_{12}\frac{\partial}{\partial x} + \Gamma^2_{12}\frac{\partial}{\partial y} - \Gamma^1_{21}\frac{\partial}{\partial x} - \Gamma^2_{21}\frac{\partial}{\partial y} = 0$$

hence

$$\Gamma^1_{12} = \Gamma^1_{21} \quad \text{and} \quad \Gamma^2_{12} = \Gamma^2_{21}$$

and the others are proved by following a similar argument

Appendix C

Codazzi's equation

To show that Codazzi's equation

$$(\nabla_X)Y = (\nabla_Y)X$$

is equivalent to ψ being holomorphic where

$$\psi = \{(a_{11} - a_{22}) - 2ia_{12}\}e^{\phi}$$

Proof :

Codazzi's equation

$$(\nabla_X)Y = (\nabla_Y)X$$

in local coordinates with $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ is

$$\nabla_{\frac{\partial}{\partial x}}(A\frac{\partial}{\partial y}) - A(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}) = \nabla_{\frac{\partial}{\partial y}}(A\frac{\partial}{\partial x}) - A(\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x})$$

Recall $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} - \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} = 0$, thus

$$\nabla_{\frac{\partial}{\partial x}}(A\frac{\partial}{\partial y}) = \nabla_{\frac{\partial}{\partial y}}(A\frac{\partial}{\partial x})$$

$$\nabla_{\frac{\partial}{\partial x}}(a_{21}\frac{\partial}{\partial x} + a_{22}\frac{\partial}{\partial y}) = \nabla_{\frac{\partial}{\partial y}}(a_{11}\frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y})$$

$$(a_{21})_x\frac{\partial}{\partial x} + a_{12}\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + (a_{22})_x\frac{\partial}{\partial y} + a_{22}\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}$$

$$= (a_{11})_y\frac{\partial}{\partial x} + a_{11}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} + (a_{12})_y\frac{\partial}{\partial y} + a_{12}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}$$

Hence as $a_{21} = a_{12}$ and $\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y}$ we have

$$\{(a_{12})_x - (a_{11})_y\} \frac{\partial}{\partial x} + \{(a_{22})_x - (a_{21})_y\} \frac{\partial}{\partial y}$$

$$= a_{12}(\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} - \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}) + (a_{11} - a_{22})(\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y})$$

$$= a_{12}\{(\Gamma^1_{22} - \Gamma^1_{11}) \frac{\partial}{\partial x} + (\Gamma^2_{22} - \Gamma^2_{11}) \frac{\partial}{\partial y}\} + \ (a_{11} - a_{22})(\Gamma^1_{12} \frac{\partial}{\partial x} + \Gamma^2_{12} \frac{\partial}{\partial y})$$

equating $\frac{\partial}{\partial y}$'s we have

$$(a_{11})_x + (a_{12})_y + a_{12}(\Gamma^2_{22} - \Gamma^2_{11}) + 2(a_{11} - H)\Gamma^2_{12} = 0$$

and equating $\frac{\partial}{\partial x}$'s we have

$$-(a_{12})_x + (a_{11})_y + a_{12}(\Gamma^1_{22} - \Gamma^1_{11}) + 2(a_{11} - H)\Gamma^1_{12} = 0$$

resulting in

$$(a_{11})_x + (a_{11} - H)\phi_x + (a_{12})_y + a_{12}\phi_y = 0$$

and

$$(a_{11})_y + (a_{11} - H)\phi_y + (a_{12})_x + a_{12}\phi_x = 0$$

multiplying by $2e^{\phi}$ and rearranging we have

$$2(a_{11})_x e^{\phi} + (2a_{11} - 2H)\phi_x e^{\phi} = -2(a_{12})_y e^{\phi} - 2a_{12}\phi_y e^{\phi}$$

and

$$2(a_{11})_y e^{\phi} + (2a_{11} - 2H)\phi_y e^{\phi} = -2(a_{12})_x e^{\phi} - 2a_{12}\phi_x e^{\phi}$$

Now $2(a_{11})_x = (2a_{11} - 2H)_x = (a_{11} - a_{22})_x$ and similarly $2(a_{11})_y = (a_{11} - a_{22})_y$, hence

$$(a_{11} - a_{22})_x e^{\phi} + (a_{11} - a_{22})\phi_x e^{\phi} = -2(a_{12})_y e^{\phi} - 2a_{12}\phi_y e^{\phi}$$

and

$$(a_{11} - a_{22})_y e^{\phi} + (a_{11} - a_{22})\phi_y e^{\phi} = 2(a_{12})_x e^{\phi} 2a_{12}\phi_x e^{\phi}$$

or similarly

$$[(a_{11} - a_{22})e^{\phi}]_x = [-2a_{12}e^{\phi}]_y$$

and

$$[(a_{11} - a_{22})e^{\phi}]_y = [2a_{12}e^{\phi}]_x$$

letting

$$u(x,y) = (a_{11} - a_{22})e^{\phi} \quad \text{and} \quad v(x,y) = -2a_{12}e^{\phi}$$

it is obvious that these are simply the Cauchy Riemann Equations for the function

$$\psi = \{(a_{11} - a_{22}) - 2\imath a_{12}\}e^{\phi}$$

It is clear that if we start with the assumption that ψ is holomorphic then we can get back to Codazzi's eqation by reversing the order of these steps

Appendix D

The Gauss Curvature

From the preliminaries we have

$$K = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

and letting $X = \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ we have

$$\begin{aligned}
R(\frac{\partial}{\partial x},\frac{\partial}{\partial y})\frac{\partial}{\partial y} &= \nabla_{\frac{\partial}{\partial x}}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} - \nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} - \nabla_{[\frac{\partial}{\partial x},\frac{\partial}{\partial y}]}\frac{\partial}{\partial y} \\
&= \nabla_{\frac{\partial}{\partial x}}\{\Gamma^1_{22}\frac{\partial}{\partial x} + \Gamma^2_{22}\frac{\partial}{\partial y}\} - \nabla_{\frac{\partial}{\partial y}}\{\Gamma^1_{12}\frac{\partial}{\partial x} + \Gamma^2_{12}\frac{\partial}{\partial y}\} \\
&= \nabla_{\frac{\partial}{\partial x}}\{-\frac{\phi_x}{2}\frac{\partial}{\partial x} + \frac{\phi_y}{2}\frac{\partial}{\partial y}\} - \nabla_{\frac{\partial}{\partial y}}\{\frac{\phi_y}{2}\frac{\partial}{\partial x} + \frac{\phi_x}{2}\frac{\partial}{\partial y}\} \\
&= -\frac{\phi_{xx}}{2}\frac{\partial}{\partial x} - \frac{\phi_x}{2}\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + \frac{\phi_{xy}}{2}\frac{\partial}{\partial y} + \frac{\phi_y}{2}\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} \\
&\quad -[\frac{\phi_{yy}}{2}\frac{\partial}{\partial x} + \frac{\phi_y}{2}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial x} + \frac{\phi_{xy}}{2}\frac{\partial}{\partial y} + \frac{\phi_x}{2}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}] \\
&= -\frac{1}{2}(\phi_{xx} + \phi_{yy})\frac{\partial}{\partial x} - \frac{\phi_x}{2}(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} + \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}) \\
&= -\frac{1}{2}(\phi_{xx} + \phi_{yy})\frac{\partial}{\partial x}
\end{aligned}$$

thus

$$\begin{aligned} K &= \frac{g(-\frac{1}{2}(\phi_{xx}+\phi_{yy})\frac{\partial}{\partial x},\frac{\partial}{\partial x})}{e^{\phi}e^{\phi}-0} \\ &= -\frac{1}{2}e^{-2\phi}\Delta\phi g(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) \\ &= -\frac{1}{2}e^{-\phi}\Delta\phi \end{aligned}$$

Appendix E

Immersion into hyperbolic 2-sphere

$\Delta = \mathbf{S}^2$

In this case we find that $f(M)$ is conformally a piece of the hyperbolic 2-sphere S_1^2

Proof ·

It is easy to see that when Δ is just a sphere, the shape operator A is simply a multiple of of the identity I Hence

$$A = \lambda I$$

for some function $\lambda = \lambda(x, y)$ Codazzi's equation implies that the function λ is just a constant and infact

$$A = \frac{1}{r} I$$

where r is the radius of the sphere Recall

$$X\xi = f_*(AX)$$

letting $X = \frac{\partial}{\partial x}$ we have

$$\xi_x = f_x$$

similarly with $X = \frac{\partial}{\partial y}$ we have

$$\xi_y = f_y$$

thus

$$\xi = f - c$$

for some $c \in \mathbb{R}^3$ Hence

$$f = \xi + c$$

i e $f(M)$ is a piece of the hyperbolic 2-sphere S_1^2

Appendix F

A differential equation

To examine solutions of the differential equations

$$f''(x) = 2\,c^2 f^3(x) \tag{F 1}$$

where $c \in \mathbf{R}$ and initial conditions

i $\quad f(x_0) = r^2 \qquad$ and $\qquad f'(x_0) = c\,r^4$

ii $\quad f(x_0) = -r^2 \qquad$ and $\qquad f'(x_0) = -c\,r^4$

Results :

(F 1) with initial conditions i gives the solution

$$f(x) = \frac{1}{c(d-x)}, \quad d = x_0 + \frac{1}{c\,r^2}$$

which is defined on the semi-infinite line $(-\infty, d)$ or (d, ∞) depending on whether c is positive of negative repectively

(F 1) with initial conditions ii gives the solution

$$f(x) = \frac{-1}{c(d-x)}, \quad d = x_0 + \frac{1}{c\,r^2}$$

which is defined on the semi-infinite line $(-\infty, d)$ or (d, ∞) again depending on whether c is positive of negative repectively

Proof of (F 1) with initial conditions i

$$
\begin{aligned}
(F\,1) \Rightarrow 2f'(x)f''(x) &= 4c^2 f^3(x) f'(x) \\
(f'(x)^2)' &= (c^2 f^4(x))' \\
f'(x)^2 &= c^2 f^4(x) + c_1 \\
f'(x_0)^2 &= c^2 f^4(x_0) + c_1 \Rightarrow c_1 = 0 \\
f'(x)^2 &= [c f^2(x)]^2 \\
f'(x) &= \pm c f^2(x) \\
f'(x_0) &= \pm c f^2(x_0) \Rightarrow \pm = + \\
f'(x) &= c f^2(x) \\
\int \frac{1}{f^2} df &= c \int dx \\
\frac{-1}{f(x)} &= c(x-d) \\
f(x) &= \frac{-1}{c(x-d)} \\
f(x_0) &= \frac{-1}{c(x_0-d)} = r^2 \Rightarrow d = x_0 + \frac{1}{cr^2}
\end{aligned}
$$

and (F 1) with initial conditions ii is proved in a similar manner

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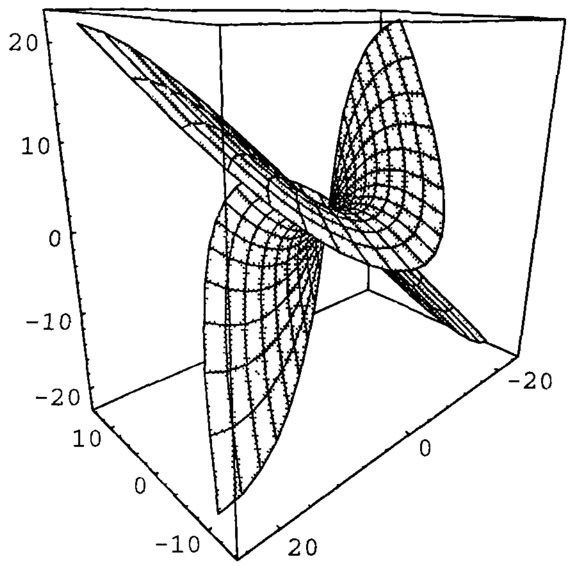
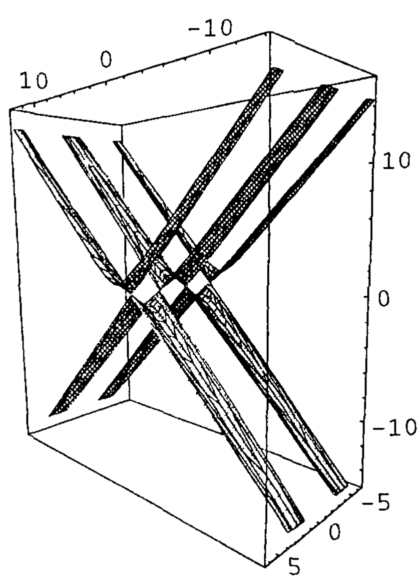
Examples of Minimal Surfaces

Theorem 1 5, equation 1 with $r_1 = 1, r_2 = 0$ and $l = 6$



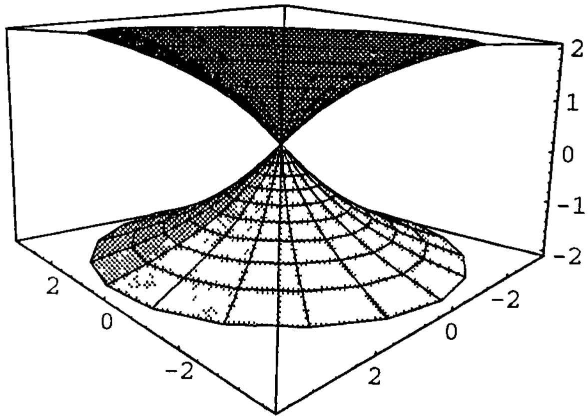
This is the surface of Enneper of the second kind, which is a minimal spacelike surface of revolution

Theorem 1 5, equation 2 with $r_1 = 1, r_2 = 0$ and $\epsilon = 1$



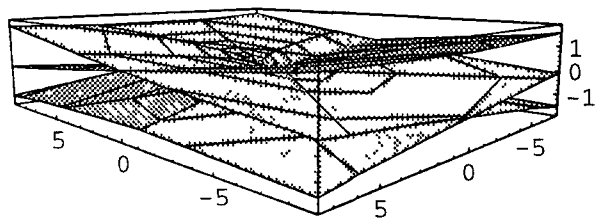
This is the catenoid of the first kind, which is a minimal surface of revolution

Theorem 1 5, equation 3 with $r_1 = -1, r_2 = 0$ and $\epsilon = -1$



This is the catenoid of the first kind, which is a minimal surface of revolution

Theorem 1 5, equation 6 with $r_1 = 2, r_2 = 4, \alpha = -2$ and $\epsilon = -1$



Theorem 1 5, equation 7, with $r_1 = 2, r_2 = -1$ and $\alpha = \frac{1}{2}$